

Discussion 10

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Absorbtion and Dispersion

If a material is conducting $J_f = \sigma E$. After transient behavior, we have:

$$\nabla \cdot E = 0, \nabla \cdot E = -\partial_t B, \nabla \cdot B = 0, \nabla \times B = \mu \epsilon \partial_t E + \mu \sigma E.$$

This gives rise to modified wave equations for E and B:

$$\nabla^2 E = \mu \epsilon \partial_t^2 E + \mu \sigma \partial_t E$$

$$\nabla^2 B = \mu \epsilon \partial_t^2 B + \mu \sigma \partial_t B$$

As presented in the last discussion this admits plane-wave solutions ($\propto e^{i(\tilde{k}z - \omega t)}$) but with a complex wave number $\tilde{k} = k + i\kappa$

$$\text{where } k = \omega \sqrt{\frac{\epsilon\mu}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2}}, \kappa = \omega \sqrt{\frac{\epsilon\mu}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}}$$

The imaginary part of \tilde{k} , κ , manifests as an attenuation of the waves:

$$\tilde{E}(z, t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{E},$$

$$\tilde{B}(z, t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} (\hat{z} \times \hat{E})$$

Things to note: the waves must be transverse due to the first two maxwells equations so $\hat{E} \cdot \hat{z} = 0$. It is common to refer to the distance it take the amplitude to attenuate by a factor of $1/e$ as the skin depth: $d = 1/\kappa$. The real part k determines the wavelength, propagation speed, and the index of refraction in the usual way: $\lambda = 2\pi/k, v = \omega/\kappa, n = c k/\omega$.

For some problems please see the first few of attached old "solutions".

Freq Dependence of Dielectric

Please read 9.4.3. It's very clearly written and provides a simple yet very instructive model for light propagating through dilute gas. The idea is that the force on an electron in the media can be described

as:

$$F_{\text{tot}} = F_{\text{binding}}(\text{spring}) + F_{\text{damping}}(\text{opposing motion}) + F_{\text{driving}}(\text{electromagnetic wave})$$

so we get a nice differential equation: $\partial_t^2 x + \gamma \partial_t x + \omega_0^2 x = \frac{q}{m} E_0 e^{-i\omega t}$. The dipole moment is the real part of the solution for x multiplied by the charge: $p = q x(t)$. Different electrons within a molecule experience different natural frequencies and damping coefficients, giving rise to a complex dielectric constant for the model:

$$\tilde{\epsilon}_r = 1 + N \frac{q^2}{m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}, \text{ where } \omega \text{ is the frequency of the wave, } f_j \text{ is the number of particles with natural frequency } \omega_j, \text{ and}$$

damping coefficient γ_j . The complex dielectric constant gives rise to dispersion and absorption of electromagnetic waves propagating through the media.

If we stay away from resonances, we can ignore the damping, and we get the following formula for the index of refraction:

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}. \quad \text{For } \omega < \omega_j \text{ we can further simplify using: } \frac{1}{\omega_j^2 - \omega^2} \approx \frac{1}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right), \text{ or}$$

$$n = 1 + A(1 + B/\lambda^2), \text{ where } A = \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \text{ is called the coefficient of refraction, and}$$

$$B = \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \text{ is called the coefficient of dispersion.}$$

How do we find the natural frequencies of particles? Well, we can consider a simple toy model presented in Example 4.1:

a primitive model for an atom consists of a point nucleus (+q), surrounded by a uniformly charged spherical cloud (-q) of radius

$$a. \text{ The electric field a distance } x \text{ away from the center but inside the cloud would be } E = \frac{1}{4\pi\epsilon_0} \frac{qx}{a^3} \implies$$

$$F = -qE = -\left(\frac{1}{4\pi\epsilon_0} \frac{q^2}{a^3}\right)x = -k_{\text{spring}}x = -m\omega_0^2x, \text{ So } \omega_0 = \sqrt{\frac{q^2}{4\pi\epsilon_0 m a^3}}.$$

This is a surprisingly decent model. If you assume a radius of the atom is .05 angstroms you get within an order of magnitude of hydrogen at 0° C and atmospheric pressure. (using $f = 2$ for H_2 , and $N = \text{avogadro's} \# / 22.4 \text{ liters} = 2.69 \times 10^{25}$).

c.f. problem 9.23

Solving Maxwell's equations.

Recall that using 4-vectors we can unify electric and magnetic potentials into a single 4-potential:

$$A^\mu = (V/c, A_x, A_y, A_z)^\mu$$

We can also unify our descriptions of charge and current densities into a single four vector: $J^\mu = (c\rho, J_x, J_y, J_z)^\mu$.

If we choose the Lorentz gauge:

$$\mu_0 \epsilon_0 \partial_t V + \nabla \cdot \vec{A} = 0 \text{ (c.f. 10.12)} \implies$$

$$\frac{1}{c^2} \partial_t V + \nabla \cdot \vec{A} = 0 \implies$$

$$\frac{\partial_t}{c} A^0 + \partial_x A^1 + \partial_y A^2 + \partial_z A^3 = 0 \implies$$

$$\partial_0 A^0 + \partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3 = 0 \implies$$

$$\partial_\mu A^\mu = 0$$

Which means our Maxwell's equations: $\partial_\nu F^{\mu\nu} = J^\mu$

$$\partial^\mu \partial_\nu A^\nu - \partial^\nu \partial_\nu A^\mu = \mu_0 J^\mu, \text{ where } (F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu),$$

reduce to the following inhomogeneous wave equations for the potential four-vector:

$$\partial_\nu \partial^\nu A^\mu = -\mu_0 J^\mu. \text{ Other ways of writing this:}$$

$$\square^2 A^\mu = -\mu_0 J^\mu$$

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2\right) A^\mu = -\mu_0 J^\mu$$

This is 4 equations: (10.16)

$$\square^2 V / c = -\mu_0 c \rho \iff \square^2 V = -\mu_0 \frac{1}{\mu_0 \epsilon_0} \rho$$

$$(\square^2) \vec{A} = -\mu_0 \vec{J}$$

This has a solution!

$$A^\mu(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(\vec{r}', t_r \equiv t - \frac{r}{c})}{r} d\tau', \text{ where } r \equiv |\vec{r} - \vec{r}'|$$

In terms of separate components we see:

$$\frac{V}{c} = \frac{\mu_0}{4\pi} \int \frac{c \rho(\vec{r}', t_r(r, r', t))}{r} d\tau' \implies$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r(r, r', t))}{r} d\tau' \text{ and } \vec{A} = \frac{\mu_0}{4\pi} \int \frac{J(r', t_r(r, r', t))}{r} d\tau'$$

To prove that this is the solution we only need to prove it in one component $A^0 = \frac{\mu_0}{4\pi} \int \frac{\rho}{r} d\tau'$ as all others are exactly of the same form, so that's our problem.

So we're going to show that $\square^2 V = -\frac{\rho}{\epsilon_0}$

First we need ∇V . (note gradient is on unprimed coords):

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[(\nabla \rho) \frac{1}{r} + \rho \nabla \left(\frac{1}{r} \right) \right] d\tau$$

$$\begin{aligned} \nabla \rho &= (\partial_{t_r} \rho) \nabla t_r \\ &= (\partial_{t_r} \rho) \nabla \left(t - \frac{1}{c} r \right) \\ &= -\frac{1}{c} (\partial_{t_r} \rho) \nabla r \end{aligned}$$

Note: $\partial_{t_r} F(t_r) = (\partial_t F(t_r)) \frac{\partial t_r}{\partial t} = (\partial_t F(t_r)) (1) = \partial_t F(t_r)$ for all functions of t_r , so we can say: $\nabla \rho = -\frac{1}{c} (\partial_t \rho) \nabla r$

Now: $\nabla r = \hat{r}$, and $\nabla \left(\frac{1}{r} \right) = -\frac{\hat{r}}{r^2}$.

$$\begin{aligned} \text{Aside: why? } \partial_i \left(\frac{1}{r} \right) &= \partial_i \left(\frac{1}{(r_j r_j + r'_j r'_j - 2 r_j r'_j)^{1/2}} \right) \\ &= -\frac{1}{2} \frac{1}{(r_j r_j + r'_j r'_j - 2 r_j r'_j)^{3/2}} (2 r_i - 2 r'_i) \\ &= -\frac{(r_i - r'_i)}{(r_j r_j + r'_j r'_j - 2 r_j r'_j)^{3/2}} \\ &= -\frac{r_i}{r^3} = -\frac{\hat{r}_i}{r^2} \end{aligned}$$

$$\implies \nabla V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{1}{c} (\partial_t \rho) \frac{\hat{r}}{r} - \rho \left(\frac{\hat{r}}{r^2} \right) \right] d\tau$$

Now we need to take the divergence. $\nabla^2 V = \nabla \cdot \nabla V$

Recall: $\nabla \cdot (f \vec{A}) = (\nabla f) \cdot \vec{A} + f (\nabla \cdot \vec{A})$ (inside cover)

$$\nabla^2 V = \nabla \cdot \nabla V$$

$$= \frac{1}{4\pi\epsilon_0} \int \left\{ -\frac{1}{c} \left[(\nabla \partial_t \rho) \cdot \frac{\hat{\mathbf{r}}}{r} + (\partial_t \rho) \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r} \right) \right] - \left[(\nabla \rho) \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) + \rho \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) \right] \right\} d\tau$$

$$\nabla \partial_t \rho = -\frac{1}{c} \partial_t^2 \rho \nabla \mathbf{r} = -\frac{1}{c} \partial_t^2 \rho \hat{\mathbf{r}}$$

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r} = \frac{1}{r^2} \text{ (problem 1.62)}$$

Aside: why? Easiest to do in spherical coordinates translating the origin to \mathbf{r}' .

The divergence is the same in both coordinate systems as it's invariant under translation.

But in our new coordinates we have only the problem: $\nabla \cdot \frac{\hat{\mathbf{r}}}{r} = \frac{1}{r^2} \partial_r (r^2 \times \frac{1}{r}) = \frac{1}{r^2}$.

Now convert back to our original coordinates and we understand: $\nabla \cdot \frac{\hat{\mathbf{r}}}{r} = \frac{1}{r^2}$.

More explicitly:

$\mathbf{r}_i = r_i - (r')_i$, so $\partial_{r_i} F(\vec{\mathbf{r}}) = \partial_{r'_i} F(\vec{\mathbf{r}}) \frac{\partial r'_i}{\partial r_i}$. So:

$$\partial_{r'_i} \frac{\hat{\mathbf{r}}_i}{r} = \partial_{r_i} \frac{\hat{\mathbf{r}}_i}{r} = \nabla_{\mathbf{r}} \cdot \frac{\hat{\mathbf{r}}}{r} = \frac{1}{r^2} \partial_r (r^2 \times \frac{1}{r}) = \frac{1}{r^2}.$$

Using spherical \mathbf{r} coordinates as above.

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi \delta^3(\vec{\mathbf{r}}) \text{ (eqn. 1.100)}$$

$$\begin{aligned} \nabla^2 V &= \frac{1}{4\pi\epsilon_0} \int \left\{ -\frac{1}{c} \left[-\frac{1}{c} \partial_t^2 \rho \hat{\mathbf{r}} \cdot \frac{\hat{\mathbf{r}}}{r} + (\partial_t \rho) \left(\frac{1}{r^2} \right) \right] - \left[-\frac{1}{c} (\partial_t \rho) \hat{\mathbf{r}} \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) + \rho (4\pi \delta^3(\vec{\mathbf{r}})) \right] \right\} d\tau \\ &= \frac{1}{4\pi\epsilon_0} \int \left\{ \left[\frac{1}{c^2} \partial_t^2 \rho \cdot \frac{1}{r} - \frac{1}{c} (\partial_t \rho) \left(\frac{1}{r^2} \right) \right] + \left[\frac{1}{c} (\partial_t \rho) \left(\frac{1}{r^2} \right) - \rho (4\pi \delta^3(\vec{\mathbf{r}})) \right] \right\} d\tau \\ &= \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{c^2} \partial_t^2 \rho \cdot \frac{1}{r} \right] d\tau - \frac{1}{4\pi\epsilon_0} \int \rho \left(\vec{\mathbf{r}}, t - \frac{\mathbf{r}}{c} \right) (4\pi \delta^3(\vec{\mathbf{r}})) d\tau \\ &= \frac{1}{c^2} \partial_t^2 [V] - \frac{1}{\epsilon_0} \int \rho(\vec{\mathbf{r}}, t - 0) (\delta^3(\vec{\mathbf{r}})) d\tau \\ &= \frac{1}{c^2} \partial_t^2 [V] - \frac{1}{\epsilon_0} \rho(\vec{\mathbf{r}}, t) \end{aligned}$$

Which we see satisfies $\square^2 V = -\frac{1}{\epsilon} \rho$.

Yeah, but you've only solved the most boring of maxwell's equations (other than $\nabla \cdot \mathbf{B} = 0$), I hear you cry.

Fear not, the same derivation shows that the starting formula is a general solution for each of the components of the four potential A^μ , which tells you everything you need to know about the fields through $F^{\mu\nu}$.

1 Griffiths, 9-19

■ a) Poor conductor

Show that the skin depth in a poor conductor ($\sigma \ll \omega \epsilon$) is $2/\sigma \sqrt{\epsilon/\mu}$ independent of frequency. Find the skin depth (in meters) for (pure) water.

We're given the definition of skin depth $d \equiv \frac{1}{\kappa}$, where $\kappa \equiv \omega \sqrt{\frac{\epsilon\mu}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right)^{1/2}}$

We recall what it means to series expand $f(a)$ about small a : $f(a) \rightarrow f(0) + f'(0)a + \frac{1}{2} f''(0)a^2 + O(a^3)$

Well, if we series expand $\sqrt{1+a^2} \rightarrow 1 + \frac{a^2}{2} + O(a^3)$.

So expanding $\sqrt{\sqrt{1+a^2} - 1} \rightarrow \frac{a}{\sqrt{2}} + O(a^3)$, or substituting in our original expression we have:

$$\kappa = \frac{1}{2} \sqrt{\epsilon\mu} \sqrt{\frac{1}{\epsilon^2\omega^2}} \omega\sigma + O(\sigma^3) = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} \sigma + O(\sigma^3)$$

So to leading order in σ , we have $d = \frac{1}{\kappa} = \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}}$

For pure water we have $\epsilon = 80.1 \epsilon_0$ (Table 4.2), $\mu = \mu_0(1 + \chi_m)$, (Table 6.1)

$\sigma = 1/(2.5 \times 10^5)$ Table (7.1), so $d = 1.19 \times 10^4$ m

■ b) Good conductor

Show that the skin depth in a good conductor ($\sigma \gg \omega \epsilon$) is $\lambda/(2\pi)$ independent of frequency. Find the skin depth (in meters) for a typical metal ($\sigma = 10^7 (\Omega m)^{-1}$) in the visible range ($\omega \approx 10^{15} /s$), assuming $\epsilon = \epsilon_0$, and $\mu = \mu_0$

Again we have $\kappa = \omega \sqrt{\frac{\epsilon\mu}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right)^{1/2}}$, except this time σ is large.

In this situation, we have $\frac{\sigma}{\epsilon\omega}$, dominating the square root, and so $\kappa \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\mu\omega\sigma}{2}}$. Similarly

$$k = \omega \sqrt{\frac{\epsilon\mu}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right)^{1/2}} \approx \sqrt{\frac{\mu\omega\sigma}{2}}, \text{ is this regime so we have } \kappa \approx k. \quad k = \frac{2\pi}{\lambda} \implies$$

$$d = \frac{1}{\kappa} \approx \frac{1}{k} = \frac{\lambda}{2\pi}.$$

So $\kappa = \sqrt{\frac{\mu\omega\sigma}{2}} = \sqrt{\frac{4\pi \times 10^{-7} \times 10^{15} \times 10^7}{2}} = 7.92665 \times 10^7 \frac{1}{\text{meter}}, d = \frac{1}{\kappa} = 1.26 \times 10^{-8}$ meters. Not very far at all.

■ c) Lag

Show that in a good conductor the magnetic field lags the electric field by $\pi/4$ radians, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part(b).

We get our lag from the phase given in eqn 9.134:

$$\phi = \arctan(\kappa/k) = \arctan(1) = \pi/4.$$

$$\text{Eqn 9.137 gives us } \frac{B_0}{E_0} = \sqrt{\frac{\sigma\mu}{\omega}} = \sqrt{\frac{10^7 \cdot 4\pi \times 10^{-7}}{10^{15}}} = 1.121 \times 10^{-7}.$$

2 Linear Approximation

$$\partial_t \mathbf{J} = -\partial_t e n \mathbf{v}_e \simeq -e n_0 \partial_t \mathbf{v} = -e n_0 \mathbf{a} = -e n_0 \left(\frac{1}{m_e} \mathbf{F} \right) = -e n_0 \left(\frac{-e}{m_e} \mathbf{E} \right) = + \frac{e^2 n_0}{m_e} \mathbf{E}.$$

Our wave equation is:

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu \epsilon \partial_t^2 \mathbf{E} + \mu \sigma \partial_t \mathbf{E} = \mu \epsilon \partial_t^2 \mathbf{E} + \mu \partial_t \mathbf{J} \\ &= \mu \epsilon \partial_t^2 \mathbf{E} + \frac{\mu e^2 n_0}{m_e} \mathbf{E} \end{aligned}$$

\Leftrightarrow

$$\left(\nabla^2 - \frac{\mu e^2 n_0}{m_e} - \mu \epsilon \partial_t^2 \right) \mathbf{E} = 0 \Leftrightarrow$$

$$\left(\frac{1}{\mu \epsilon} \nabla^2 - \frac{e^2 n_0}{\epsilon m_e} - \partial_t^2 \right) \mathbf{E} = 0 \Leftrightarrow$$

$$(c^2 \nabla^2 - \omega_p^2 - \partial_t^2) \mathbf{E} = 0, \text{ where } \omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e}.$$

■ b

Substitute $\mathbf{E} \propto \exp[i(kz - \omega t)]$ to find the dispersion relation $\omega(k)$ associated with this wave equation:

$$\text{Plugging in we find: } -c^2 e^{i(kz - \omega t)} k^2 - e^{i(kz - \omega t)} \omega_p^2 + e^{i(kz - \omega t)} \omega^2 = 0 \Leftrightarrow$$

$$-c^2 k^2 + \omega^2 - \omega_p^2 = 0, \text{ or } \omega = \sqrt{c^2 k^2 + \omega_p^2}$$

■ c

For copper, the conduction electron density is approximately $n_0 \simeq 8.5 \times 10^{22} \text{ cm}^{-3}$

Find ω_p . Well, using:

$$e \rightarrow 1.60218 \times 10^{-19} \text{ Coulomb}$$

$$n_0 \rightarrow \frac{8.5 \times 10^{22}}{\text{Meter}^3}$$

$$\epsilon_0 \rightarrow \frac{8.85419 \times 10^{-12} \text{ Ampere Second}}{\text{Meter Volt}}$$

$$m_e \rightarrow 9.10938 \times 10^{-31} \text{ Kilogram}$$

we find:

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e} = 2.70522 \times 10^{32} \frac{1}{\text{second}^2} \Rightarrow$$

$$\omega_p = 1.64475 \times 10^{16} \frac{1}{\text{second}}$$

■ d

Discuss the attenuation of an electromagnetic wave with $\lambda = 700 \text{ nm}$ impinging on a copper mirror.

My original (sloppy) interpretation:

Well, $\lambda = \frac{2\pi}{k}$, and for a good conductor we have: $k \approx \kappa$, so $\kappa \approx \frac{2\pi}{\lambda}$, and the wave attenuates as: $e^{-\kappa z} = e^{-\frac{2\pi z}{\lambda} \times 10^7}$, if z is given in meters. So it drops off pretty quickly. (i.e. it reaches half of its initial amplitude in: 7.72225×10^{-8} meters of material)

What the professor is looking for:

The professor's wants us to explore the attenuation due to propagation below cutoff in a plasma, $\omega < \omega_p$, (λ_p is 114 nm, and λ is 700 nm). So the attenuation constant is given by $\kappa = \sqrt{\omega_p^2 - \omega^2} / c = 5.48252 \times 10^7 \text{ m}^{-1}$

in contrast to simply $\frac{2\pi}{\lambda} = 8.97598 \times 10^6 \text{ m}^{-1}$, which is off by an order of magnitude.

■ e

What is the phase velocity of a soft X-ray with $\lambda = 5 \times 10^{-9} \text{ m}$ in copper? v_g ?

$$\omega = \sqrt{c^2 k^2 + \omega_p^2} \Rightarrow$$

$$\begin{aligned} \frac{\omega}{k} &= \sqrt{c^2 + \frac{\omega_p^2}{k^2}} = c \sqrt{1 + \frac{\omega_p^2}{k^2 c^2}} \\ &= c \sqrt{1 + \frac{2.70522 \times 10^{32}}{(2\pi/5 \times 10^{-9})^2 8.98755 \times 10^{16}}} \text{ m/s} \\ &= 1.00095 c \end{aligned}$$

We calculate the group velocity:

$$\frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{c^2 k^2 + \omega_p^2}} = c \times \frac{k}{\sqrt{k^2 + \frac{\omega_p^2}{c^2}}} = 0.999048 c$$

scratch

3 Griffiths, 9.29

Confirm that the energy in the TE_{mn} mode travels at the group velocity.

Hint: Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle \mu \rangle$ (use Prob 9.11).

Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave and take their ratio.

Well, from problem 9.11, we see that:

$$\langle u \rangle = \frac{1}{4} \left(\epsilon_0 \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* + \frac{1}{\mu_0} \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}}^* \right) \text{ and } \langle \mathbf{S} \rangle = \frac{1}{2\mu_0} (\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*)$$

As we're going to be integrating over the cross section the only relevant component from $\langle \mathbf{S} \rangle$ is $\langle S \rangle_z$.

What does a wave in a waveguide look like? It looks like 9.180:

$$E_x = \frac{i}{(\omega/c)^2 - k^2} (k \partial_x E_z + \omega \partial_y B_z)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2} (k \partial_y E_z - \omega \partial_x B_z)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2} (k \partial_x B_z - \frac{\omega}{c^2} \partial_y E_z)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} (k \partial_y B_z + \frac{\omega}{c^2} \partial_x E_z)$$

Ahh, but what about a TE_{mn} mode? TE means $E_z = 0$, and we're given

$B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/a)$ from equation 9.186. Plugging in:

$$E_x = \frac{i\omega}{(\omega/c)^2 - k^2} \left(\frac{-n\pi}{b} B_0 \cos(m\pi x/a) \sin(n\pi y/b) \right)$$

$$E_y = \frac{-i\omega}{(\omega/c)^2 - k^2} \left(\frac{-m\pi}{a} B_0 \sin(m\pi x/a) \cos(n\pi y/b) \right)$$

$$B_x = \frac{ik}{(\omega/c)^2 - k^2} \left(\frac{-m\pi}{a} B_0 \sin(m\pi x/a) \cos(n\pi y/b) \right)$$

$$B_y = \frac{ik}{(\omega/c)^2 - k^2} \left(\frac{-n\pi}{b} B_0 \cos(m\pi x/a) \sin(n\pi y/b) \right)$$

The rest is simply algebra. Just realize that the integral is over $d\mathbf{a}$, whose direction is \hat{z} .

$$\begin{aligned} \langle S \rangle_z &= \frac{1}{2\mu_0} \epsilon_{zij} \tilde{E}_i \tilde{B}_j^* = \frac{1}{2\mu_0} (\tilde{E}_x \tilde{B}_y^* - \tilde{E}_y \tilde{B}_x^*) \\ &= \frac{B_0^2 \pi^2}{2\mu_0} \left(\frac{\omega k}{((\omega/c)^2 - k^2)^2} \right) \left(\left(\frac{n}{b} \right)^2 \cos(m\pi x/a)^2 \sin(n\pi y/a)^2 + \left(\frac{m}{a} \right)^2 \sin(m\pi x/a)^2 \cos(n\pi y/b)^2 \right) \end{aligned}$$

Integrating we find:

$$\int \langle S \rangle_z d\mathbf{a} = \frac{B_0^2 \pi^2}{2\mu_0} \left(\frac{\omega k}{(\omega/c)^2 - k^2} \right) \left(\left(\frac{n}{b} \right)^2 \frac{ab}{4} + \left(\frac{m}{a} \right)^2 \frac{ab}{4} \right)$$

Now to calculate $\langle u \rangle = \frac{1}{4} \left(\epsilon_0 \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* + \frac{1}{\mu_0} \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}}^* \right)$:

$$\begin{aligned} \langle \mu \rangle &= \frac{1}{4} \left(\epsilon_0 \frac{\omega^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0^2 \left(\left(\frac{n}{b} \right)^2 \cos^2 \left(\frac{m\pi x}{a} \right) \sin^2 \left(\frac{n\pi y}{b} \right) + \left(\frac{m}{a} \right)^2 \sin^2 \left(\frac{m\pi x}{a} \right) \cos^2 \left(\frac{n\pi y}{b} \right) \right) + \right. \\ &\quad \frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0^2 \left(\left(\frac{m}{a} \right)^2 \sin^2 \left(m\pi x/a \right) \cos^2 \left(n\pi y/b \right) + \left(\frac{n}{b} \right)^2 \cos^2 \left(m\pi x/a \right) \sin^2 \left(n\pi y/b \right) \right) + \\ &\quad \left. \frac{1}{\mu_0} B_0^2 \cos^2 \left(\frac{m\pi x}{a} \right) \cos^2 \left(\frac{n\pi y}{b} \right) \right) \end{aligned}$$

Integrating we find:

$$\int \langle u \rangle dA = \frac{ab}{16} \left\{ \epsilon_0 \frac{\omega^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0^2 \left(\left(\frac{n}{b} \right)^2 + \left(\frac{m}{a} \right)^2 \right) + \frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega/c)^2 - k^2)^2} \left(\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right) + \frac{B_0^2}{\mu_0} \right\}$$

We can use $((\omega/c)^2 - k^2) = (\omega_{mn}/c)^2$, $\frac{1}{c^2 \mu_0} = \epsilon_0$, and $((\frac{m}{a})^2 + (\frac{n}{b})^2) = (\frac{\omega_{mn}}{\pi c})^2$, to clean these up a bit:

$$\begin{aligned} \int \langle S \rangle_z dA &= \frac{B_0^2 \pi^2}{2 \mu_0} \left(\frac{\omega k}{((\omega/c)^2 - k^2)^2} \right) \left(\left(\frac{n}{b} \right)^2 \frac{ab}{4} + \left(\frac{m}{a} \right)^2 \frac{ab}{4} \right) = \frac{B_0^2 \pi^2}{2 \mu_0} \left(\frac{\omega k}{(\omega_{mn}/c)^4} \right) \left(\frac{\omega_{mn}}{\pi c} \right)^2 \frac{ab}{4} \\ &= \frac{B_0^2}{8 \mu_0} \frac{ab c^2 \omega k}{\omega_{mn}^2} \end{aligned}$$

and (using $(kc)^2 = (\omega)^2 - (\omega_{mn})^2$)

$$\begin{aligned} \int \langle u \rangle dA &= \frac{ab}{16} \left\{ \frac{1}{c^2 \mu_0} \frac{\omega^2 \pi^2}{((\omega_{mn}/c)^2)^2} B_0^2 \left(\left(\frac{\omega_{mn}}{\pi c} \right)^2 \right) + \frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega_{mn}/c)^2)^2} \left(\left(\frac{\omega_{mn}}{\pi c} \right)^2 \right) + \frac{B_0^2}{\mu_0} \right\} \\ &= \frac{ab}{16} \frac{B_0^2}{\mu_0} \left(\frac{\omega^2}{(\omega_{mn})^2} + \frac{k^2 c^2}{(\omega_{mn})^2} + 1 \right) \\ &= \frac{ab}{16} \frac{B_0^2}{\mu_0 (\omega_{mn})^2} (\omega^2 + (\omega)^2 - (\omega_{mn})^2 + \omega_{mn}^2) = \frac{ab B_0^2 \omega^2}{8 \mu_0 (\omega_{mn})^2} \end{aligned}$$

Great. So now we've got:

$$\frac{\int \langle S \rangle_z dA}{\int \langle u \rangle dA} = \frac{B_0^2}{8 \mu_0} \frac{ab c^2 \omega k}{\omega_{mn}^2} \bigg/ \frac{ab B_0^2 \omega^2}{8 \mu_0 (\omega_{mn})^2} = \frac{c^2 k}{\omega} = \frac{c}{\omega} \sqrt{\omega^2 - \omega_{mn}^2} = v_g$$

4 TEM Waves

In section 9.5.3, it is shown that the TEM wave travels with both phase and group velocity c in a coaxial conductor system.

■ a

Show that the electric and magnetic fields in the coaxial system arise from a charge per unit length and current that are related by c .

We're given that $E_s = \frac{A}{s} \cos(kz - \omega t)$, and $B_\phi = \frac{A}{c s} \cos(kz - \omega t)$

We can use Gauss's law to find the charge enclosed, specifically for a cylinder of radius s and length dz :

$$\oint E \cdot da = \frac{Q_{enc}}{\epsilon_0} \iff 2\pi s \times E_s dz = \frac{1}{\epsilon_0} \lambda dz \implies \lambda = \epsilon_0 2\pi s E_s = \epsilon_0 2\pi A \cos(kz - \omega t)$$

We can use Ampere's law to find the current enclosed, specifically for a circle of radius s :

$$\oint B \cdot dl = \mu_0 I_{enc} \implies 2\pi s B_\phi = \mu_0 I_{enc} \implies I = \frac{1}{\mu_0} \frac{2\pi}{c} A \cos(kz - \omega t) = \frac{\lambda}{c \mu_0 \epsilon_0} = c \lambda$$

■ b

What is the phase and group velocity in the case that the conductors have a dielectric of frequency independent permittivity ϵ between them?

We're really playing a two dimensional game here, and we can simply say that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, and so we shift c to $\frac{1}{\sqrt{\mu_0 \epsilon}}$ in the dispersion relation: $k = \sqrt{\mu_0 \epsilon} \omega$. In this case $v = \frac{1}{\sqrt{\mu_0 \epsilon}}$, as does v_g .

If one actually takes the cartesian laplacian on the solutions above we find that the dispersion relation is really that simple. i.e. These solutions have to satisfy $\nabla^2 E_i = \mu_0 \epsilon \partial_t^2 E_i$. But for the TEM modes as given above (only x and y components when you translate to cartesian), we see that only the z derivative survives. i.e.:

$$E_x = E_s \cos(\phi) = E_s \frac{x}{s} = \frac{Ax}{(x^2+y^2)} \cos(kz - \omega t)$$

$$\begin{aligned} \text{Note: } (\partial_x^2 + \partial_y^2) \frac{x}{(x^2+y^2)} &= \left(\frac{8x^2}{(x^2+y^2)^3} - \frac{2}{(x^2+y^2)^2} \right) x + \left(\frac{8y^2}{(x^2+y^2)^3} - \frac{2}{(x^2+y^2)^2} \right) x - \frac{4x}{(x^2+y^2)^2} \\ &= \frac{8x^3}{x^6+3y^2x^4+3y^4x^2+y^6} - \frac{8x}{x^4+2y^2x^2+y^4} + \frac{8y^2x}{x^6+3y^2x^4+3y^4x^2+y^6} \\ &= 0 \end{aligned}$$

So we find: $\partial_z^2 E_x = \mu_0 \epsilon \partial_t^2 E_x \iff k^2 = \mu_0 \epsilon \omega^2$ which is the simple dispersion relation above.

The same holds for E_y , B_x , and B_y .

■ c

What is the relationship between the voltage between the two conductors and the current I ?

$$\begin{aligned} E_s &= \frac{\lambda}{\epsilon_0 2\pi s} \implies \\ \Delta V &= -\int_a^b E_s ds = \frac{\lambda}{\epsilon 2\pi} \ln\left(\frac{a}{b}\right) = \frac{\sqrt{\epsilon \mu_0} I}{\epsilon 2\pi} \ln\left(\frac{a}{b}\right) = \sqrt{\frac{\epsilon \mu_0}{\epsilon^2}} \frac{I}{2\pi} \ln\left(\frac{a}{b}\right) \end{aligned}$$

$$\text{So the line impedance } Z = \frac{\Delta V}{I} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon}} \ln\left(\frac{a}{b}\right).$$

For a dielectric loaded coaxial guide with $\epsilon = 3$, can we pick a ratio a/b to make this impedance 50 Ω ?

Well, the ratio has to be:

$$\frac{a}{b} = e^{2\pi z \sqrt{\frac{\epsilon}{\mu_0}}}$$

If we really mean $\epsilon = 3 \times \epsilon_0$, then: $\frac{a}{b} = 4.2392$.

If $\epsilon \equiv \chi_e$, then $\epsilon = (1 + \chi_e) = 4 \times \epsilon_0$ and $\frac{a}{b} = 5.3006$

And if $\epsilon = 3 \frac{\text{Ampere Second}}{\text{Meter Volt}}$ (i.e. 3 \times SI units for permittivity), then $\frac{a}{b} = 2.264309697 \times 10^{210809}$, which is unlikely.

5 Resonant Cavities

Consider the resonant cavity produced by closing off the two ends of a rectangular wave guide, at $z = 0$, and $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by:

$$\omega_{lmn} = c\pi \sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$

Let's look at the boundary conditions: $E^{\parallel} = 0$, and $B^{\text{perp}} = 0$. Also, as we've no stuff inside,

$$\nabla \cdot E = 0.$$

$E^{\parallel} = 0$, means that:

$$E_y(a, y, z) = E_y(0, y, z) = E_z(a, y, z) = E_z(0, y, z) = 0$$

$$E_x(x, b, z) = E_x(x, 0, z) = E_z(x, b, z) = E_z(x, 0, z) = 0$$

$$E_x(x, y, d) = E_x(x, y, 0) = E_y(x, y, d) = E_y(x, y, 0) = 0$$

$$\text{Maxwell's equations: } \nabla \times E = i\omega B, \nabla \times B = -\frac{i\omega}{c^2} E \implies$$

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = \nabla \times (i\omega B) = \frac{\omega^2}{c^2} E$$

Note: we need only consider the E field as the B field can be arrived at through $\nabla \times E$.

$$\text{Our differential equation is: } \nabla^2 E_i = -\frac{\omega^2}{c^2} E_i$$

For example: $\nabla^2 E_x = -\frac{\omega^2}{c^2} E_x$. Let: $E_x(x, y, z) = X(x)Y(y)Z(z)$. Separation of variables suggests, $\partial_i^2 R_i(r_i) = -k_i^2 R_i(r_i)$, for each component. Like so:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) E_x = -\frac{\omega^2}{c^2} E_x \implies (\partial_x^2 + \partial_y^2 + \partial_z^2) X(x)Y(y)Z(z) = -\frac{\omega^2}{c^2} X(x)Y(y)Z(z)$$

$$\implies (\partial_x^2 X(x)Y(y)Z(z) + \partial_y^2 X(x)Y(y)Z(z) + \partial_z^2 X(x)Y(y)Z(z)) = -\frac{\omega^2}{c^2} X(x)Y(y)Z(z)$$

$$\implies (\partial_x^2 X(x))Y(y)Z(z) + X(x)Z(z)(\partial_y^2 Y(y)) + X(x)Y(y)\partial_z^2(Z(z)) = -\frac{\omega^2}{c^2} X(x)Y(y)Z(z)$$

Now divide both sides by E_x :

$$\implies \frac{(\partial_x^2 X(x))}{X(x)} + \frac{(\partial_y^2 Y(y))}{Y(y)} + \frac{\partial_z^2(Z(z))}{Z(z)} = -\frac{\omega^2}{c^2}$$

So now, we let each independent expression equal a constant (eg. $\frac{(\partial_x^2 X(x))}{X(x)} = -k_x^2$) such that the sum of the constants is $-\omega^2/c^2$:

$$-k_x^2 - k_y^2 - k_z^2 = -\frac{\omega^2}{c^2}$$

So now we have the following differential equations:

$$\partial_x^2 X(x) = -k_x^2 X(x), \partial_y^2 Y(y) = -k_y^2 Y(y), \partial_z^2 Z(z) = -k_z^2 Z(z)$$

These solutions are sinusoids, and our parallel boundary condition removes cosine terms from the components parallel to the appropriate boundaries:

i.e we get something of the form:

$$E = A \cos(k_x x) \sin(k_y y) \sin(k_z z) \hat{x} + B \sin(k_x x) \cos(k_y y) \sin(k_z z) \hat{y} + C \sin(k_x x) \sin(k_y y) \cos(k_z z) \hat{z}$$

Where $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, and $k_z = \frac{l\pi z}{d}$ are required for the sin terms to vanish appropriately.

Note: only cosine terms survive on component (i.e. we get a $\cos(k_z z)$ for the \hat{z} component, a $\cos(k_x x)$ for the \hat{x} component, etc) due to the requirement of $\nabla \cdot E = 0$. Why? $\partial_z \cos(k_z z) \propto \sin(k_z z)$, which vanishes at the boundary.

As the sum over k_i^2 must equal $(\frac{\omega}{c})^2$ from our separation of variables, we have $\pi \sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2} = \frac{\omega_{\text{mnl}}}{c^2}$, and we're done.

6 Midterm problem

Consider a circular current loop located in the $z = 0$ plane, tracing out a radius a about the origin, and sinusoidal current excitation $I = I_0 \sin(\omega t)$.

■ a)

What is the time dependent magnetic dipole moment due to this current loop?

$$m = I \times \text{area}^2 = I_0 \sin(\omega t) \pi a^2 \hat{z}$$

■ b)

Given the magnetic vector potential find the electric field.

$$\text{Well: } \nabla \times E = -\partial_t B = -\partial_t (\nabla \times A) = -\nabla \times \partial_t A$$

So modulo functions mapping to zero under the curl (i.e. gradients of scalar fields etc) :

$$\begin{aligned} E &= -\partial_t A(t) \\ &= -\frac{\mu_0}{4} a^2 \frac{I_0 \omega \cos(\omega t)}{r^3} (\hat{z} \times \vec{r} = r \sin(\theta) \hat{\phi}) \\ &= -\frac{\mu_0}{4} a^2 I_0 \omega \cos(\omega t) \frac{s}{r^3} \hat{\phi} \end{aligned}$$