Review TE modes

Find the TE_{20}, and TE_{11} mode for a rectangular wave guide, and in doing so refresh our waveguide knowledge.

Let's use Griffith's convention that the wave guide is along the z direction.

What does a wave in a waveguide look like? It looks like eqn 9.180:

$$\tilde{E}_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_x E_z + \omega\,\partial_y B_z\right) e^{i(k z - \omega t)}$$
$$\tilde{E}_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_y E_z - \omega\,\partial_x B_z\right) e^{i(k z - \omega t)}$$
$$\tilde{B}_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_x B_z - \frac{\omega}{c^2}\,\partial_y E_z\right) e^{i(k z - \omega t)}$$
$$\tilde{B}_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_y B_z + \frac{\omega}{c^2}\,\partial_x E_z\right) e^{i(k z - \omega t)}$$

where E_z and B_z satisfy (9.181):

$$\left(\partial_x{}^2 + \partial_y{}^2 + (\omega/c)^2 - k^2\right)\tilde{E}_z = 0$$
$$\left(\partial_x{}^2 + \partial_y{}^2 + (\omega/c)^2 - k^2\right)\tilde{B}_z = 0$$

A TE mode has $E_z = 0$, a TM mode has $B_z = 0$, a TEM wave has $E_z = B_z = 0$, which can not happen in a hollow wave guide.

Ok, with $E_z = 0$, we can solve the remaining differential equation with ease using seperation of variables:

$$\left(\partial_x{}^2 + \partial_y{}^2 + (\omega/c)^2 - k^2\right)\tilde{B}_z = 0$$

we assume $\tilde{B}_z = X(x)\,Y(y)\,e^{i(k z - \omega t)}$, plug in, and divide by $\tilde{B}_z$:

$$\frac{1}{\tilde{B}_z}\left(\partial_x{}^2 + \partial_y{}^2 + (\omega/c)^2 - k^2\right) X(x)\,Y(y)\,e^{i(k z - \omega t)} = 0 \Longrightarrow$$
$$\frac{1}{\tilde{B}_z}\left[Y\left(\partial_x{}^2\right)X + X\left(\partial_y{}^2\right)Y + X\,Y\left(+(\omega/c)^2 - k^2\right)\right] e^{i(k z - \omega t)} = 0 \Longrightarrow$$
$$\left(\frac{1}{X}\left(\partial_x{}^2 X\right) + \frac{1}{Y}\left(\partial_y{}^2 Y\right) + (\omega/c)^2 - k^2\right) = 0$$

This can be satisfied with:

$$\frac{1}{X}\left(\partial_x{}^2 X\right) = -k_x{}^2$$
$$\frac{1}{Y}\left(\partial_y{}^2 Y\right) = -k_y{}^2$$

such that: $\left(-k_x{}^2 + -k_y{}^2 + (\omega/c)^2 - k^2\right) = 0$. This is a natural form for the dispersion relation, we'll see how to express the dispersion relation in terms of the mode variables m and n. in a second.

Well, the general solution to the X equation is:
$X(x) = A\sin(k_x x) + B\cos(k_x x)$

But the boundary conditions require that $B_x = 0$ (perpendicular component of B vanishes at each boundary: 9.175, similarly parallel component of E field vanishes at each boundary). Since $B_x \propto \partial_x B_z$ for the TE mode (see 9.180 above) then
we must have $\partial_x B_z = (\partial_x X)\,Y\,e^{i(\mathrm{kz} - \omega t)} = 0 \Longleftrightarrow$
$\partial_x X(x) = 0$. The boundaries are at $x = 0$, and $x = a$, so
we need $k_x = m\pi / a$ ($m = 0, 1, 2, 3$ ....), and $A = 0$. Similarly for Y with: $k_y = n\pi / b$ ($n = 0, 1, 2, 3$ ....).

Our final $B_z = B_0 \cos(m\pi x / a)\cos(n\pi y / b)\,e^{i(k z - \omega t)}$

and we can rewrite our dispersion relation from:
$\left(-k_x^2 + -k_y^2 + (\omega/c)^2 - k^2\right) = 0$ to
$\left(-\left(\frac{m\pi}{a}\right)^2 + -\left(\frac{n\pi}{b}\right)^2 + (\omega/c)^2 - k^2\right) = 0$

The wavenumber can be expressed:
$k = \sqrt{(\omega/c)^2 - \pi^2\left[(m/a)^2 + (n/b)^2\right]}$
$= \sqrt{(\omega/c)^2 - (\omega_{mn}/c)^2}$

The cutoff frequency $\omega_{mn}{}^2 \equiv c^2\left((m\pi/a)^2 + (n\pi/b)^2\right)$.

For frequencies less then this, the wave number will be imaginary, and will therefore exponentially attenuate.

We can again write the dispersion relation:
$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega_{mn}}{c}\right)^2$

In any case: $\tilde{B}_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)\, e^{i(kz-\omega t)}$ for a TE mode (where a and b are the heighth and width of the waveguide, for a square guide these two are equivalent: $a = b = \ell$. I'll use this convention to simplify notation for the rest.)

So, this B_z satisfies the following boundary conditions:

So with $\tilde{B}_z = B_0 \cos(m\pi x/\ell)\cos(n\pi y/\ell)\, e^{i(kz-\omega t)}$, we can plug into 9.180 above to find all components of the wave.

$\tilde{E}_x = \frac{i\omega}{(\omega/c)^2-k^2}\left(-\frac{n\pi}{\ell} B_0 \cos(m\pi x/\ell)\sin(n\pi y/\ell)\right) e^{i(kz-\omega t)}$
$\tilde{E}_y = -\frac{i\omega}{(\omega/c)^2-k^2}\left(-\frac{m\pi}{\ell} B_0 \sin(m\pi x/\ell)\cos(n\pi y/\ell)\right) e^{i(kz-\omega t)}$
$\tilde{B}_x = \frac{ik}{(\omega/c)^2-k^2}\left(-\frac{m\pi}{b} B_0 \sin(m\pi x/\ell)\cos(n\pi y/\ell)\right) e^{i(kz-\omega t)}$
$\tilde{B}_y = \frac{ik}{(\omega/c)^2-k^2}\left(-\frac{n\pi}{b} B_0 \cos(m\pi x/\ell)\sin(n\pi y/\ell)\right) e^{i(kz-\omega t)}$

or using $(\omega/c)^2 - k^2 = (\omega_{mn}/c)^2$, we can simplify the above.

We can see that at $x = 0$ and ℓ, $B_x = 0$ for all m Similarly, for $y = 0$ and ℓ, $B_y = 0$ for all n so our boundary conditions are satisfied for B. Similarly the parallel components vanish for E $\left(E_y(x, y)\,|_{x=0,\,x=\ell} = 0\ ,\ E_x(x, y)\,|_{y=0,\,y=\ell} = 0\right)$. All of our boundary conditions are met.

The mode TE_{20}, is the above with $m = 2, n = 0$.
The real components of the fields would be:
$E_x = 0$
$E_y = \frac{\omega}{(\omega_{20}/c)^2}\left(-\frac{2\pi}{\ell} B_0 \sin(2\pi x/\ell)\right)\sin(kz - \omega t)$
$E_z = 0$
$B_x = -\frac{k}{(\omega_{20}/c)^2}\left(-\frac{2\pi}{\ell} B_0 \sin(2\pi x/\ell)\right)\sin(kz - \omega t)$
$B_y = 0$
$B_z = B_0 \cos\left(\frac{2\pi x}{\ell}\right)\cos(kz - \omega t)$

The mode TE_{11},on the other hand, with $m = 1, n = 1$.
The real components of the fields would be:
$\tilde{E}_x = -\frac{\omega}{(\omega_{11}/c)^2}\left(-\frac{\pi}{\ell} B_0 \cos(\pi x/\ell)\sin(\pi y/\ell)\right)\sin(kz - \omega t)$

$E_y = \frac{\omega}{(\omega_{11}/c)^2}\left(-\frac{\pi}{\ell} B_0 \sin(\pi x/\ell)\cos(\pi y/\ell)\right)\sin(k z - \omega t)$
$E_z = 0$
$B_x = -\frac{k}{(\omega_{11}/c)^2}\left(-\frac{\pi}{\ell} B_0 \sin(\pi x/\ell)\cos(\pi y/\ell)\right)\sin(k z - \omega t)$
$B_y = 0$
$B_z = B_0 \cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi y}{\ell}\right)\cos(k z - \omega t)$

Nifty.

Ok, all that's left is to talk about the wave velocity:
$v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$, and the group velocity:
$v_g = \frac{1}{dg/d\omega} = c\sqrt{1-(\omega_{mn}/\omega)^2}$

There's an excellent picture: fig 9.25, that visualizes these results in a way that could be useful on the exam. I recommend reading the discussion relating to this figure, and putting it in context of what we just discussed. You can see $\sqrt{1-(\omega_{mn}/\omega)^2}$ as the cosine of an angle to z ($\cos(\theta)$) that an ordinary plane wave is propagating on in the waveguide, reflecting off of all walls.

Now what happens if I close off my walls? Pretty much exactly the same solution holds, except my z component has to satisfy boundary conditions, and is therefore "quantized" like the x and y components.
c.f. problem 9.38. (look at problem 5 at the end of discussion 10).

Review EM fields in different reference frames.

An electromagnetic plane wave of (angular) frequency ω is traveling in the x direction through the vacuum. It is polarized in the y direction, and the amplitude of the electric field is E_0

(a) Write down the electric and magnetic fields $E(x, y, z, t)$,
$B(x, y, z, t)$. Define any auxiliary quantities we introduce in terms of ω E_0 and the constants of nature (heh)

Eqn 9.48 has E pointing in the x direction, and B pointing in the y direction, and the EM field going in the z direction:

$E(z, t) = E_0 \cos(k z - \omega t + \delta)\,\hat{x}$

$B(z, t) = \frac{1}{c} E_0 \cos(k z - \omega t + \delta)\,\hat{y}$

For our problem we just change variables, and set the phase (δ) to 0:

$E(x, y, z, t) = E_0 \cos(k x - \omega t)\,\hat{y}$

$B(x, y, z, t) = \frac{1}{c} E_0 \cos(k x - \omega t)\,\hat{z}$, with $k = \omega / c$

(b) The same wave is observed from an inertial system $\overline{S}$ moving in the x direction with speed v relative to the original system $\mathcal{S}$. Find the electric and magnetic fields in $\overline{S}$, and express them in terms of the $\tilde{S}$ coordinates. $\overline{E}(\overline{x}, \overline{y}, \bar{z}, \bar{t}), \overline{B}(\overline{x}, \overline{y}, \bar{z}, \bar{t})$

First we can use the inverse lorentz transforms to get how the coordinates change:

$x = \gamma(\overline{x} + v\,\bar{t})$, and $t = \gamma\left(\bar{t} + \frac{v}{c^2}\,\overline{x}\right)$

so:

$k\,x - \omega\,t = k\,\gamma\left(\bar{x} + v\,\bar{t}\right) - \omega\,\gamma\left(\bar{t} + \frac{v}{c^2}\,\bar{x}\right)$

$= \bar{x}\left(k\,\gamma - \omega\,\gamma\,\frac{v}{c^2}\right) - \bar{t}\,(\omega\,\gamma - k\,\gamma\,v)$

$= \bar{x}\,\bar{k} - \bar{t}\,\bar{\omega}$

where (using that k=ω/c):
$\bar{k} = \gamma\left(k - \omega\,\frac{v}{c^2}\right) = \gamma\,k\,(1 - v/c) = \alpha\,k$
$\bar{\omega} = \gamma(\omega - k\,v) = \gamma\,\omega\,(1 - v/c) = \alpha\,\omega$

where $\alpha \equiv \gamma(1 - v/c) = \sqrt{\frac{1 - v/c}{1 + v/c}}$

We use Eq 12.108 to transform the fields:

$\overline{E_x} = \overline{E_z} = 0$
$\overline{E_y} = \gamma\left(E_y - v\,B_z\right) = \gamma\,E_0\left(\cos(k\,x - \omega\,t) - \frac{v}{c}\cos(k\,x - \omega\,t)\right)$
$= \alpha\,E_0\cos(k\,x - \omega\,t)$

$\overline{B_x} = \overline{B_y} = 0$
$\overline{B_z} = \gamma\left(B_z - \frac{v}{c^2}\,E_y\right) = \gamma\,E_0\left(\frac{1}{c}\cos(k\,x - \omega\,t) - \frac{v}{c^2}\cos(k\,x - \omega\,t)\right)$
$= \alpha\,\frac{1}{c}\,E_0\cos(k\,x - \omega\,t)$

stick in the new coordinates and defining $\overline{E_0} = \alpha\,E_0$, we have:
$\overline{E}\left(\bar{x}, \bar{y}, \bar{z}, \bar{t}\right) = \overline{E_0}\cos\left(\bar{k}\,\bar{x} - \bar{\omega}\,\bar{t}\right)\hat{y}$
$\overline{B}\left(\bar{x}, \bar{y}, \bar{z}, \bar{t}\right) = \frac{1}{c}\,\overline{E_0}\cos\left(\bar{k}\,\bar{x} - \bar{\omega}\,\bar{t}\right)\hat{z}$

Awesome, now let's think a second about potentials

$A^\mu = \left(0, 0, \frac{E_0}{\omega}\sin(k\,x - \omega\,t), 0\right)^\mu$ reproduces the above fields.

proof:

$\vec{E} = -\nabla A^0 - \partial_t A^i\,\hat{x}_i = +\,E_0\cos(k\,x - \omega\,t)\left(\hat{x}_2 \equiv \hat{y}\right)$

$\vec{B} = \epsilon_{\text{ijk}}\,\partial_j A_k\,\hat{x}_i = \hat{z}\,\frac{k}{\omega}\,E_0\cos(k\,x - \omega\,t) = \hat{z}\,E_0/c\cos(k\,x - \omega\,t)$

Now let's consider the potential expressed in the "barred" variables:

$A^\mu\left(\bar{x}, \bar{y}, \bar{z}, \bar{t}\right) = \left(0, 0, \frac{E_0}{\omega}\sin\left(\bar{x}\,\bar{k} - \bar{t}\,\bar{\omega}\right), 0\right)^\mu$

We lorentz transform to get $\overline{A}^\mu$:

$\overline{A}^\mu = (\Lambda^\mu{}_\nu)\,A^\nu$

$\overline{A}^\mu = \left(0, 0, \frac{E_0}{\omega}\sin\left(\bar{k}\,\bar{x} - \bar{t}\,\bar{\omega}\right), 0\right)^\mu$

Calculating $\overline{\vec{E}} = -\partial_t \overline{A}_y \, \hat{y}$

$= E_0 \frac{\overline{\omega}}{\omega} \cos\left(\overline{k}\,\overline{x} - \overline{t}\,\overline{\omega}\right) \hat{y}$

$= \overline{E_0} \cos\left(\overline{k}\,\overline{x} - \overline{\omega}\,\overline{t}\right) \hat{y}$

and $\overline{\vec{B}} = \hat{\overline{z}}\, \partial_{\overline{x}} \overline{A}_y = \frac{\overline{k}}{\omega} E_0 \cos\left(\overline{k}\,\overline{x} - \overline{\omega}\,\overline{t}\right) \hat{\overline{z}}$

$= \overline{E_0} \cos\left(\overline{k}\,\overline{x} - \overline{\omega}\,\overline{t}\right) \hat{\overline{z}}$

same as before.