# Ph 114. Assignment 3

## Solutions
## TA: John Joseph M. Carrasco
## jjmc@physics.ucla.edu

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### 5.14.1

a) If c is a function only of z, show that $d\theta/d$s=- $(\cos(\theta_0)/c_0)\, dc/dz$, with $\theta_0$ the angle of elevation of the ray where $c = c_0$.

We know (c.f. eqn 5.14.15) that $\frac{d}{ds}\left(\left(n \equiv \frac{c_0}{c}\right)\times \cos(\theta)\right) = 0 \Longrightarrow \frac{c_0}{c}\cos(\theta) = \text{ constant } = \frac{c_0}{c(z=0)=c_0}\cos(\theta(z=0)=\theta_0)$
$\Longrightarrow \frac{\cos(\theta)}{c} = \frac{\cos(\theta_0)}{c_0}$.

Expanding the differential equation, $\frac{d}{ds}\left(\left(n \equiv \frac{c_0}{c}\right)\times \cos(\theta)\right) = 0$, also gives us the relation:
$0 = \frac{d}{ds}(n\cos(\theta)) = \left(\frac{dn}{ds}\right)\cos(\theta) - n\sin(\theta)\frac{d\theta}{ds} \Longrightarrow \frac{dn}{ds} = n\frac{\sin(\theta)}{\cos(\theta)}\frac{d\theta}{ds}$

We also know that $\frac{d}{ds}(n\sin(\theta)) = \frac{dn}{dz} \Longrightarrow$
$\left(\frac{d}{ds}n\right)\sin(\theta) + n\cos(\theta)\frac{d\theta}{ds} = \frac{dn}{dz}$ , substituing in for $\frac{dn}{ds}$ from above, and recalling $\sin^2 + \cos^2 = 1$ we have:

$$\frac{d\theta}{ds}\left(\frac{n}{\cos(\theta)}\right) = \frac{dn}{dz} = -\frac{c_0}{c^2}\frac{dc}{dz} \Longrightarrow$$

$\frac{d\theta}{ds} = -\frac{c_0}{c^2}\frac{c}{c_0}\cos(\theta)\frac{dc}{dz} = -\frac{\cos(\theta)}{c}\frac{dc}{dz} = -\frac{\cos(\theta_0)}{c_0}\frac{dc}{dz}$, and we're done.

b) If g = dc / dz is a constant, find the radius of curvature R. Is R a constant?

looking at picture 15.14.2 in the text, dz=sin($\theta$) ds. Defining R≡ $\left|\frac{ds}{d\theta}\right|$, and plugging in from above, we find:
$R = \left|\left(-\frac{\cos(\theta_0)}{c_0}\frac{dc}{dz}\right)^{-1}\right| = \left|\left(-\frac{\cos(\theta_0)}{c_0}g\right)^{-1}\right| = \left|\frac{c_0}{\cos(\theta)}\frac{1}{g}\right|$.

Yes, R is constant.

c) If the temperature of air decreases linearly with height z, verify that $c(z) = c_0 - g\,z$. The speed of sound goes linearly with temperature ⟹ if T is linear with z, so must c. $c(z) = c_0 + k\,z$, we find out what k is by differentiating giving us: k=$\frac{dc}{dz} = g$.

Now they ask us to find radius of curvature for $\theta_0 = 0$, and to ask at what x, z will have reached 10 meters.

From the link off my web-page, we find an equation for speed of sound as a function of temperature:
$c(z) = c_0\, m/s + .6\, m/s/C * T(z)$

If $T(z) = -5\,C/\text{km} * (z)$, then we have:
$c(z) = 340 \;\; + .6\;(-5\,z) = 340 - 3\,z \implies |g|=3 \implies \boxed{R = \frac{c_0}{g} \; = \; 113.33}$

Now: $\frac{\text{dz}}{\text{dx}} = \tan(\theta) \implies \int \text{dx} = \int \text{dz}\,\frac{1}{\tan(\theta(z))} = \int \text{dz}\;1 \Big/ \sqrt{\frac{1}{\eta^2 c^2} - 1}$

$$= \int \text{dz}\;1 \Big/ \sqrt{\frac{340^2}{(340-3\,z)^2} - 1} \quad \text{(see 14.2.}a\text{ for details)}$$

We solve this integrating dz from 0 km, to .01 km, giving us $\boxed{\int \text{dx} = 1.5\,\text{km}}$.

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## 5.14.2

Assume the speed of sound is given by the quasi-parobolic profile $c(z) = c_0\left(1 - (\epsilon\,z)^2\right)^{-1/2}$. Let the depth z=0, which defines the axis of the sound chanel, lie well below the surface of the ocean.

### ■ a) Find the equation z(x) for rays emitted by a source at (x,z)=(0,0) with angles of elevation or depression $\pm\theta_0$

Hint: use Snell's law, $\frac{\text{dz}}{\text{dx}} = \tan(\theta)$, and $\int (a^2 - u^2)^{-1/2}\,\text{du} = \sin^{-1}(u/a)$

$\frac{dz}{dx} = \tan(\theta) \implies \frac{dz}{\tan(\theta)} = d\text{x} \implies \int dx = \int dz\;1/\tan(\theta(z))$

We know that from Snell's law: $\cos(\theta(z)) = \frac{\cos(\theta_0)}{c_0}\text{c(z)} \implies \theta\text{(z)= ArcCos}\left[\frac{\cos(\theta_0)}{c_0}\;c(z)\right]$.
Let's define $\eta = \frac{\cos(\theta_0)}{c_0}$. As $\tan(\arccos(u)) = \frac{\sqrt{1-u^2}}{u}$, we have:

$\tan(\theta(z)) = \tan(\arccos(\eta\,c(z))) = \frac{\sqrt{1-\eta^2\,c^2}}{\eta\,c}$

$= \sqrt{\frac{1}{\eta^2\,c^2} - 1}$

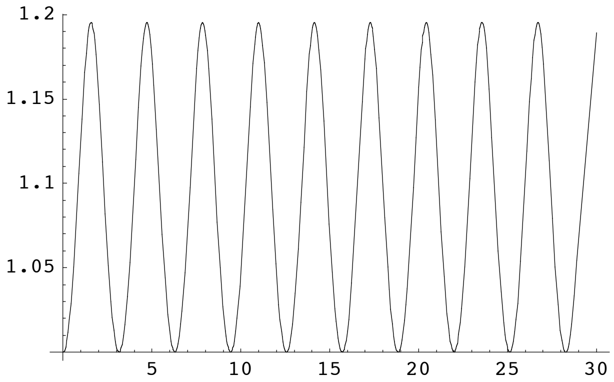
$= \sqrt{\frac{1}{\eta^2\,c_0{}^2\left(1-(\epsilon\,z)^2\right)^{-1}} - 1}$

$= \sqrt{\frac{1-(\epsilon\,z)^2}{\eta^2\,c_0{}^2} - 1} = \frac{\epsilon}{\eta\,c_0}\sqrt{\frac{1-\eta^2\,c_0{}^2}{\epsilon^2} - (z)^2}$

defining $\mu = \frac{\epsilon^2}{\eta\,c_0} = \frac{\epsilon}{\cos(\theta_0)}$, and $\alpha^2 = \frac{1-\eta^2\,c_0{}^2}{\epsilon^2} = \frac{\sin^2(\theta_0)}{\epsilon^2}$, we have:
$\tan(\theta(z)) = \mu\,\sqrt{\alpha^2 - z^2} \implies \int \frac{dz}{\tan(\theta(z))} = x \iff$

So we have: $\frac{1}{\mu}\int dz\,\frac{1}{\sqrt{\alpha^2 - z^2}} = x \implies \frac{1}{\mu}\arcsin\left(\frac{z}{\alpha}\right) = x ==> z = \frac{\sin(\theta_0)}{\epsilon}\,\sin\left(\frac{\epsilon}{\cos(\theta_0)}\;x\right)$.

Nice, huh?



### b) Average speed

This is going to be: $\frac{\Delta x}{\Delta t}$, where $\Delta t = \int_{s(x=0)}^{s(x=\Delta x)} \frac{1}{c}\, ds = \int_{x=0}^{x=\Delta x} \frac{1}{c(z(x))\cos(\theta(x))}\, dx$. This holds as: $\frac{dx}{ds} = \cos(\theta) \Longrightarrow ds = \frac{dx}{\cos(\theta)}$. I had trouble getting this integral to work out myself to cancel epsilons, but if you have the basic idea, you'll get credit.

### c) Show that for $|\theta_0| \leq \pi/8$, c(z) is a good aproximation to the parabolic profile $c_0\left(1+\frac{1}{2}(\epsilon z)^2\right)$

This follows via series expansion. Small $\theta_0 \Longrightarrow$ small change in speed of sound. can approx z as $s\sin(\theta_0) \simeq s\,\theta_0$.

$$c_0\left(1+\frac{1}{2}(\epsilon z)^2\right)^{-1/2} \cong c_0\left(1-(\epsilon s\,\theta_0)^2\right)^{-1/2} = c_0 + \frac{1}{2} s^2 \epsilon^2 c_0 \theta_0^2 + O(\theta_0^4) = c_0\left(1+\frac{1}{2}(\epsilon s\,\theta_0)^2\right) + O({\theta_0}^4)$$
$$\simeq c_0\left(1+\frac{1}{2}(\epsilon z)^2\right)$$

Rest is plugging in numbers.

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## 5.14.3

$c(z) = c_0(1-\epsilon\,|z|)^{-1/2}$

### a) Find the equation z(x) for rays emitted by a source at (x,z)=(0,0) with angles of elevation or depression $\pm\theta_0$

Hint: use Snell's law, $\frac{dz}{dx} = \tan(\theta)$

$$\frac{dz}{dx} = \tan(\theta) \Longrightarrow \frac{dz}{\tan(\theta)} = dx \Longrightarrow \int dx = \int dz\, 1/\tan(\theta(z))$$

We know that from Snell's law: $\cos(\theta(z)) = \frac{\cos(\theta_0)}{c_0}\, c(z) \Longrightarrow \theta(z) = \text{ArcCos}\left[\frac{\cos(\theta_0)}{c_0}\, c(z)\right]$.
Let's define $\eta = \frac{\cos(\theta_0)}{c_0}$. As $\tan(\arccos(u)) = \frac{\sqrt{1-u^2}}{u}$, we have:

$$\tan(\theta(z)) = \tan(\arccos(\eta\, c(z))) = \frac{\sqrt{1-\eta^2\, c^2}}{\eta\, c}$$

$$= \sqrt{\frac{1}{\eta^2\, c^2} - 1}$$

$$= \sqrt{\frac{1}{\eta^2\, {c_0}^2\, (1-(\epsilon\, |z|))^{-1}} - 1}$$

$$= \sqrt{\frac{1-\epsilon\, |z|}{\eta^2\, {c_0}^2} - 1} = \frac{\sqrt{\epsilon}}{\eta\, c_0} \sqrt{\frac{1-\eta^2\, {c_0}^2}{\epsilon} - \left| z \right|}$$

defining $\mu = \frac{\sqrt{\epsilon}}{\eta\, c_0} = \frac{\sqrt{\epsilon}}{\cos(\theta_0)}$ , and $\alpha = \frac{1-\eta^2\, {c_0}^2}{\epsilon} = \frac{\sin(\theta_0)^2}{\epsilon}$, we have:

$\tan(\theta(z)) = \mu \sqrt{\alpha - | z |} \Longrightarrow \int \frac{dz}{\tan(\theta(z))} = x$

So we have: $\frac{1}{\mu} \int_0^{\pm\beta} dz' \frac{1}{\sqrt{\alpha-|z|}} = x \Longrightarrow \mp \frac{2\left(\sqrt{\alpha \mp z} - \sqrt{\alpha}\,\right)}{\mu} = x ==> z = \alpha \pm \left(\mp \frac{x\, u}{2} + \sqrt{\alpha}\right)^2$

$$z = \alpha \pm \left(\mp \frac{x\, u}{2} + \sqrt{\alpha}\right)^2 = \alpha \pm \left( \frac{x^2\, u^2}{4} + \alpha \mp x\, u \sqrt{\alpha}\right)$$

$$= \frac{\sin(\theta_0)^2}{\epsilon} \pm \left( \frac{x^2 \left(\frac{\sqrt{\epsilon}}{\cos(\theta_0)}\right)^2}{4} + \frac{\sin(\theta_0)^2}{\epsilon} \mp x \frac{\sqrt{\epsilon}}{\cos(\theta_0)} \sqrt{\frac{\sin(\theta_0)^2}{\epsilon}} \right)$$

$$= \frac{\sin(\theta_0)^2}{\epsilon} \pm \left( x^2 \left(\frac{\epsilon}{4 \cos^2(\theta_0)}\right) + \frac{\sin(\theta_0)^2}{\epsilon} \mp x \frac{\sin(\theta_0)}{\cos(\theta_0)}\right)$$

If we take the minus, then we get:

$$z = x \frac{\sin(\theta_0)}{\cos(\theta_0)} - \frac{\epsilon\, x^2}{4 \cos^2(\theta_0)}$$

Which matches the answer in the back of the book:

Their answer: Z= 2 X - $X^2$ $\frac{\epsilon\, z}{\sin^2 \theta_0} = 2\left(\frac{\epsilon\, x}{2\, \mathrm{Sin}[\theta_0]\, \mathrm{Cos}[\theta_0]}\right) - \left(\frac{\epsilon\, x}{2\, \mathrm{Sin}[\theta_0]\, \mathrm{Cos}[\theta_0]}\right)^2$

$\Longrightarrow z = x \frac{\sin(\theta_0)}{\cos(\theta_0)} - \frac{\epsilon\, x^2}{4 \cos^2 \theta_0}$

### ■ b

For a ray with initial angle $\theta_0$, find the distance Δx between x-axis intercepts and the maximum distance Δz it attains above of below z=0.

So x-axis intercepts occur when z=0, so:

$z = x \frac{\sin(\theta_0)}{\cos(\theta_0)} - \frac{\epsilon\, x^2}{4 \cos^2 \theta_0} = 0 \Longrightarrow$ x =0 and x= 4 $\frac{\cos(\theta_0) \sin(\theta_0)}{\epsilon}$. so Δx = 4 $\frac{\cos(\theta_0) \sin(\theta_0)}{\epsilon}$

Max dist Δz:

$z' = \frac{\sin(\theta_0)}{\cos(\theta_0)} - \frac{\epsilon\, x}{2 \cos^2(\theta_0)} == 0 \Longleftrightarrow x = \frac{2 \cos(\theta_0) \sin(\theta_0)}{\epsilon} \Longrightarrow$

$z_{\mathrm{max}} = \frac{2 \cos(\theta_0) \sin(\theta_0)}{\epsilon} \frac{\sin(\theta_0)}{\cos(\theta_0)} - \frac{\epsilon}{4 \cos^2 \theta_0} \left(\frac{2 \cos(\theta_0) \sin(\theta_0)}{\epsilon}\right)^2$

$= 2 \sin^2(\theta_0) / \epsilon - \frac{4 \sin^2(\theta_0)}{4\, \epsilon} = \frac{\sin^2(\theta_0)}{\epsilon}$

### ■ c

For a given ray find an expression for the average speed with which energy propagates out to a distance x lying on the channel axis.

This is going to be : $\frac{\Delta \mathrm{x}}{\Delta \mathrm{t}}$, where $\Delta \mathrm{t} = \int_{x=0}^{x=\Delta \mathrm{x}} \frac{1}{c \cos(\theta)}\, dx$. *I* had trouble getting this integral to work out myself, but if you have the basic idea, you' ll get credit.

### ■ d

For $|\theta_0| \leq \pi/8$ show that c(z) is a good approx of the linear profile $c_0(1 + \frac{1}{2} \epsilon |z|)$.

This follows via series expansion. Small $\theta_0 \Longrightarrow$ small change in speed of sound. can approx z as $s \sin(\theta_0) \simeq s\, \theta_0$.
$c_0(1 - (\epsilon\, |z|))^{-1/2} \cong c_0(1 - (\epsilon\, s\, |\theta_0|))^{-1/2} = c_0 + \frac{1}{2}\, s\, \epsilon\, c_0\, \theta_0 + O(\theta_0^2) = c_0(1 + \frac{1}{2}\, (\epsilon\, s\, \theta_0)) + O({\theta_0}^2)$
$\simeq c_0(1 + \frac{1}{2} \epsilon |z|)$
Rest is plugging in numbers.