Euclidean Continuation of the Dirac Fermion

Mayank R. Mehta[(a)]
Department of Physics, Indian Institute of Science, Bangalore 560 012
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We have found a one-complex-parameter family of Dirac actions which interpolates between the Minkowski and a Euclidean Dirac action. The interpolating action is invariant under the "interpolating Lorentz transformations." The resultant Euclidean action is Hermitian and SO(4) invariant. There is no doubling of degrees of freedom of the Dirac fermion and no contradiction between the SO(4) invariance and the Hermiticity property of the Euclidean propagator. The Euclidean theory so obtained also satisfies the Osterwalder-Schrader positivity condition.

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For a variety of purposes, such as regularization of the path integrals, nonperturbative calculations, and lattice-gauge-theory calculations, Minkowski-space theories are continued to Euclidean space. The procedure for Euclidean continuation is expected to be continuous.

It was shown in Ref. 1 that when a theory involving fermions is continued to Euclidean space the degrees of freedom of a Dirac fermion have to be doubled so as to avoid a contradiction between Euclidean covariance of the fields and the form of the two-point function. Hence the usual continuation procedure is not continuous. Further, the Euclidean action so obtained does not have any definite Hermiticity.

In this paper we show that it is not necessary to double the degrees of freedom of a fermion when continued to Euclidean space. We present a one-parameter family of Dirac actions which interpolates between the Minkowski and a Euclidean action. For all the allowed values of the interpolating parameter the action is invariant under the "interpolating Lorentz transformations." The resultant Euclidean action is SO(4) invariant and Hermitian. The procedure for Euclidean continuation is continuous and the degrees of freedom of a Dirac fermion are not doubled. Contrary to the assertion in Ref. 1, there is no contradiction between the covariance of the fields and the form of the two-point function.

In Sec. (I) the usual continuation procedure is briefly sketched and some difficulties associated with it are pointed out. In Sec. (II) the new continuation procedure is presented. The Osterwalder-Schrader positivity of the lattice action so obtained is discussed in Sec. (III). Finally, the conclusions and some interesting features of the new continuation procedure are given in Sec. (IV).

(I) The usual continuation procedure.—The Dirac action in Minkowski space is given by

$$S^M=\int d^4x\,\psi^\dagger\gamma_0(i\not\partial-m)\psi\,. \tag{1}$$

Here,

$$g_{\mu\nu}=(+,-,-,-),\quad \{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}\,,\qquad \gamma_5^\dagger=\gamma_5,\quad \Sigma_{\mu\nu}=\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]\,. \tag{2}$$

The action (1) is invariant under Lorentz transformations, which act on the fields as

$$\psi'(x')=T(\omega)\psi(x)\Longrightarrow\bar\psi'(x')=\bar\psi(x)[T(\omega)]^{-1}\,, \tag{3}$$

where $\bar\psi=\psi^\dagger\gamma_0$ and $T(\omega)=\exp(\Sigma_{\mu\nu}\omega^{\mu\nu})$. The usual procedure for continuation involves replacing x_0 by $-ix_0$, which leads to the following Euclidean metric and γ matrices:

$$g^E_{\mu\nu}=(-,-,-,-)\,,\quad \gamma^E_\mu=-(\gamma^E_\mu)^\dagger\,,\qquad \Sigma^E_{\mu\nu}=\tfrac{1}{4}[\gamma^E_\mu,\gamma^E_\nu]\,. \tag{4}$$

Note that, in Euclidean space, under SO(4) transformations,

$$T^E(\omega)=\exp(\Sigma^E_{\mu\nu}\omega^{\mu\nu})\,,\quad \psi'(x')=T^E(\omega)\psi(x)\,, \tag{5}$$

however,

$$\psi'^\dagger(x')\gamma_0\neq\psi^\dagger(x)\gamma_0[T^E(\omega)]^{-1}\,. \tag{6}$$

Therefore the Euclidean action obtained from (1) by a mere replacement of x_0 by $-ix_0$ is not SO(4) invariant. This happens because of the extra γ_0 in the action after $\psi^\dagger$. Dropping this γ_0 leads to a Euclidean action

$$S^E=\int d^4x\,\psi^\dagger(i\not\partial^E-m)\psi\,, \tag{7}$$

which is SO(4) invariant but leads to the following Euclidean propagator:[1]

$$\langle\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle=\frac{1}{(2\pi)^4}\int d^4p\,(\not p^E-m)^{-1}_{\alpha\beta}\times\exp[-ip\cdot(x-x')]\,. \tag{8}$$

Here the left-hand side is Hermitian (with $x\to x'$) but the right-hand side has no definite Hermiticity. To avoid this difficulty it was proposed in Ref. 1 that the fermionic degrees of freedom should be doubled when going to Euclidean space. The SO(4)-invariant Euclidean action given in Ref. 1 is

$$S^E=\int d^4x\,\psi_1(i\not\partial^E-m)\psi_2\,, \tag{9}$$

with ψ_1 transforming as the inverse of ψ_2 under SO(4).

They are independent fermions and the Fourier expansion of each of them contains *eight* independent Grassmann variables,[1] compared with *four* in Minkowski space.[2] Thus there are eight independent states for an electron in Euclidean space, in contrast to four in Minkowski space, which is difficult to understand physically. Also, because of doubling the degrees of freedom, this procedure for going Euclidean is not continuous. Hence, a procedure for Euclidean continuation is needed which avoids these problems.

(II) The new continuation procedure.— Let us now consider a different procedure for Euclidean continuation. Instead of continuing the time, we directly continue the metric. Let

$$g^{\theta}_{\mu\nu}=(\cos 2\theta/|\cos 2\theta|,-,-,-)\quad(0\le\theta\le\pi/2\,,\ \theta\neq\pi/4) \tag{10}$$

be the interpolating metric between the Minkowski and the Euclidean metrics. Here θ is the interpolation parameter. The noncompact group SO(3,1) goes over to the compact group SO(4) at $\theta=\pi/4$. Hence, if θ is restricted to take real values, the metric changes discontinuously at this point. Therefore θ is allowed to take all the complex values, except $\pi/4$, between 0 and $\pi/2$. We now define an interpolating action invariant under the interpolating Lorentz transformations for all the allowed real values of θ. To do so, we define the interpolating gamma matrices γ^{θ}_{μ} as

$$\gamma^{\theta}_{0}=\frac{1}{|\cos 2\theta|^{1/2}}(\gamma_0\cos\theta+i\gamma_5\sin\theta)\,,\quad \gamma^{\theta}_{i}=\gamma_i\,,$$
$$\gamma^{\theta}_{5}=\frac{1}{|\cos 2\theta|^{1/2}}(\gamma_5\cos\theta-i\gamma_0\sin\theta)\,,\quad \Sigma^{\theta}_{\mu\nu}=\tfrac{1}{4}[\gamma^{\theta}_{\mu},\gamma^{\theta}_{\nu}]\,, \tag{11}$$

where γ_μ and γ_5 are the Minkowski-space gamma matrices, defined in Eqs. (2). These γ matrices satisfy the following relations:

$$\{\gamma^{\theta}_{\mu},\gamma^{\theta}_{\nu}\}=2g^{\theta}_{\mu\nu}\,,\quad \{\gamma^{\theta}_{\mu},\gamma^{\theta}_{5}\}=0\,. \tag{12}$$

The interpolating action is defined as

$$S^{\theta}=\int d^4x(-g^{\theta})^{1/2}\psi^{\dagger}M[i(g^{\theta})^{\mu\nu}\gamma^{\theta}_{\mu}\partial_\nu-m]\psi\,. \tag{13}$$

Here $g^{\theta}=\det g^{\theta}_{\mu\nu}$ and the matrix

$$M=\gamma_0 \tag{14}$$

can be thought of as a "metric" for contracting the spinors. It satisfies the relation

$$(\gamma^{\theta}_{\mu})^{\dagger}M=M(\gamma^{(\theta^*)}_{\mu})\,. \tag{15}$$

Using (11) and (15) we see that the fields transform as follows under the interpolating Lorentz transformations:

$$\psi'(x')=T^{\theta}(\omega)\psi(x)\Longrightarrow\psi'^{\dagger}(x')M=\psi^{\dagger}(x)M[T^{\theta}(\omega)]^{-1}\,, \tag{16}$$

where

$$T^{\theta}(\omega)=\exp(\Sigma^{\theta}_{\mu\nu}\omega^{\mu\nu})\,.$$

(When θ takes complex values, x_μ are also allowed to take complex values. Transformations of x_μ depend on θ and not on θ^*. Since the fields are functions of x_μ only and not of x^*_μ, their transformations as well will depend on θ and not on θ^*.) Using Eqs. (11), (12), and (15) we see that the action (13) is invariant under the transformation (16) for all the allowed values of the interpolating parameter θ. However, it is sufficient for our purpose to have an invariant action for $\theta=0$ and $\pi/2$. For real values of θ these transformations form a group, which is SO(3,1) for $\theta<\pi/4$ and SO(4) for $\theta>\pi/4$. Thus, we have a well-defined invariant action for a continuous set of values of θ, $0\le\theta\le\pi/2$.

Evidently, for $\theta=0$ the action (13) reduces to the Minkowski-space action (1). For $\theta=\pi/2$ we have

$$S^{\pi/2}=iS^E=i\int d^4x\,\psi^{\dagger}M[i(-i\gamma_5\partial_0-\gamma_i\partial_i)-m]\psi\,. \tag{17}$$

Let us choose the anti-Hermitian Euclidean γ matrices to be the following:

$$\gamma^E_0=i\gamma_5,\gamma^E_i=\gamma_i,\gamma^E_5=-i\gamma_0=-(\gamma^E_5)^{\dagger}\Longrightarrow M=\gamma_0=i\gamma^E_5\,. \tag{18}$$

In terms of (4) and (18) the Euclidean action in (17) can be written as

$$S^E=\int d^4x\,\psi^{\dagger}(i\gamma^E_5)(i\partial\!\!\!/^E-m)\psi\,. \tag{19}$$

Since the action (13), for arbitrary θ, is invariant under the interpolating Lorentz transformations given by (16), it follows that the Euclidean action (19) is invariant under the SO(4) transformations given by (5). This can also be seen directly: $\psi^{\dagger}$ transforms as the inverse of ψ under the SO(4) transformations and the matrix γ^E_5 commutes with $\Sigma^E_{\mu\nu}$. In fact, without $i\gamma^E_5$ the action (19) will not be Hermitian.

The propagator for this Euclidean action can be found directly[2] by functional differentiation of the Euclidean partition function. The Minkowski-space vacuum-to-vacuum amplitude in the presence of the sources χ and $\chi^{\dagger}\gamma_0$ is given by

$$W[\chi,\chi^{\dagger}\gamma_0]=\int\mathcal{D}(\psi^{\dagger}\gamma_0)\mathcal{D}\psi\exp\left[i\left(S+\int d^4x(\chi^{\dagger}\gamma_0\psi+\psi^{\dagger}\gamma_0\chi)\right)\right].$$

When continued to Euclidean space the measure $\mathcal{D}(\psi^{\dagger}\gamma_0)\mathcal{D}\psi$ in Minkowski space goes over to $\mathcal{D}(\psi^{\dagger}i\gamma^E_5)\mathcal{D}\psi$ in Euclidean space. Thus the Euclidean partition function is

$$W^E[\chi,\chi^{\dagger}i\gamma^E_5]=\int\mathcal{D}(\psi^{\dagger}i\gamma^E_5)\mathcal{D}\psi\exp\left[S^E+\int d^4x(\chi^{\dagger}i\gamma^E_5\psi+\psi^{\dagger}i\gamma^E_5\chi)\right]. \tag{20}$$

Here S^E is the Euclidean action (19). Differentiating with respect to the sources χ and $\chi^\dagger i\gamma_5^E$ yields[2]

$$\langle\psi_\alpha(x)[\psi^\dagger(x')i\gamma_5^E]_\beta\rangle=\int\frac{d^4p}{(2\pi)^2}\frac{(\not{p}^E+m)_{\alpha\beta}}{p^2-m^2}\exp[-ip\cdot(x-x')]\,. \tag{21}$$

(Here $p^2=-p_\mu p_\mu$.) Multiplying both sides by $(i\gamma_5^E)_{\beta\gamma}$ shows that both the left-hand side and the right-hand side are Hermitian. Thus, without doubling the degrees of freedom of the Dirac fermion, we have the correct Euclidean propagator, which is SO(4) invariant. Furthermore, because of the presence of γ_5^E, the apparent contradiction mentioned in Ref. 1, and discussed earlier, is avoided.

Clearly, a partition function, and hence a propagator, can also be defined for all the interpolating values of θ.

The matrix M in Eq. (14) is numerically equal to γ_0 for all the values of θ. However, since the coefficient of ∂_0 changes from γ_0 in Minkowski space to $-i\gamma_5$ in Euclidean space, the "metric" M for contracting fermions effectively changes from γ_0 to $i\gamma_5^E$.

In Minkowski space, the Hermitian conjugate of the fermions belonging to the $(0,\frac{1}{2})$ representation of the Lorentz group should be contracted with those belonging to the $(\frac{1}{2},0)$ representation, and vice versa, to form SO(3,1) invariants. However, in Euclidean space, the Hermitian conjugate of the fermions belonging to the $(0,\frac{1}{2})$ representation of SO(4) should be contracted with those belonging to the $(0,\frac{1}{2})$ representation, to form SO(4) invariants. Since γ_0 connects different helicities but γ_5 does not, the above-mentioned change in M from γ_0 to $i\gamma_5^E$ is precisely the one that leads to the correct invariants in Minkowski and Euclidean space and avoids the contradiction discussed in Ref. 1.

Thus, starting from an SO(3,1)-invariant action (1) we have arrived at an SO(4)-invariant action (19) along a continuous path. There is no doubling of degrees of freedom of the Dirac fermion and the Euclidean action is Hermitian. The procedure developed here generalizes easily to arbitrary (even) dimensions and to theories with arbitrary vector couplings. The Euclidean continuation of theories in odd spacetime dimensions, of theories with axial-vector couplings, and of Majorana and Weyl fermions, will be presented elsewhere.[3]

The interpolating action (13) has the following discrete symmetries. Parity:

$$\mathcal{P}x_0=x_0\,,\quad \mathcal{P}x_i=-x_i\,,\quad \mathcal{P}\psi=\gamma_0^\theta\psi\,,$$

$$\mathcal{P}[\psi^\dagger\cdots\psi]=\frac{\cos 2\theta}{|\cos 2\theta|}[(\mathcal{P}\psi^\dagger)\cdots(\mathcal{P}\psi)]\,;$$

time reversal:

$$\mathcal{T}x_0=-x_0\,,\quad \mathcal{T}x_i=-x_i\,,\quad \mathcal{T}z=z^*\,,\quad \mathcal{T}\theta=\theta\,,$$

$$\mathcal{T}\psi=\gamma_5\gamma_2\gamma_0^\theta\psi\,,$$

$$\mathcal{T}[\psi^\dagger\cdots\psi]=\frac{\cos 2\theta}{|\cos 2\theta|}[(\mathcal{T}\psi^\dagger)\cdots(\mathcal{T}\psi)]\,;$$

and charge conjugation:

$$\mathcal{C}x=x\,,\quad \mathcal{C}\psi=[\gamma_0\gamma_2\gamma_0^\theta]^*\psi\,,$$

$$\mathcal{C}[\psi^\dagger\cdots\psi]=\frac{\cos 2\theta}{|\cos 2\theta|}[(\mathcal{C}\psi^\dagger)\cdots(\mathcal{C}\psi)]\,;$$

where z is a complex number, and for the last two equations we have chosen a basis for the γ matrices (2) such that γ_2 is imaginary and all the other γ matrices are real. These equations reduce to the usual definitions of discrete symmetries in Minkowski space[2] and Euclidean space,[1,4,5] for $\theta=0$ and $\theta=\pi/2$, respectively.

It is interesting to note that the Euclidean action (19) is invariant under the following chiral transformation:

$$\psi\to\psi'=\exp(i\gamma_5^E\alpha)\psi\Rightarrow\psi^\dagger\to\psi'^\dagger=\psi'^\dagger\exp(i\gamma_5^E\alpha)\,. \tag{22}$$

Since γ_5^E is anti-Hermitian, the Euclidean chiral transformation (22) is a chiral *scale* transformation, unlike the Minkowskian chiral *phase* transformation. This fact leads to a better understanding of the chiral anomalies.[6]

(III) The Osterwalder-Schrader positivity.—Apart from the SO(4) invariance, a Euclidean theory must also satisfy the Osterwalder-Schrader (OS) positivity condition.[4,5] The OS positivity for gauge-invariant operators on a lattice was proved by Osterwalder and Seiler (by reflecting with respect to a place cutting the time links in half, i.e., for only "odd" separations).[5] Their proof of the OS positivity can be taken over for the lattice action obtained using our procedure as follows.

When the independent fermions ψ_1 and ψ_2 in the Euclidean action (9), obtained using the usual continuation procedure, are replaced by $\psi^\dagger(i\gamma_5^E)$ and ψ, the resultant Euclidean action is the one obtained by our continuation procedure, given by (19). The fermionic part of a lattice gauge theory action is a gauged and the lattice version of (9).[5] Hence, making the above replacement in the usual lattice action leads to the lattice action obtained using our procedure. Thus, the fermionic part of the lattice action used in Ref. 5 becomes

$$A_\Lambda^F=\sum_{x\in\Lambda}\{m\psi^\dagger(x)(i\gamma_5^E)\psi(x)-\tfrac{1}{2}\psi^\dagger(x)(i\gamma_5^E)i\gamma_\mu^E[U(g_{x,x-e_\mu})\psi(x-e_\mu)-U^\dagger(g_{x,x+e_\mu})\psi(x+e_\mu)]\}\,. \tag{23}$$

[The notation used here is slightly different from that of Ref. 6 where the Euclidean metric is $\delta_{\mu\nu}$ and $(\gamma_\mu^E)^\dagger=+\gamma_\mu^E$.] It is worth noting that unlike the usual lattice action, the above action is Hermitian.

The OS reflection operator Θ ($\Theta = \mathcal{CT}$) is defined as

$$\begin{aligned} &\Theta(x^0,x^1,x^2,x^3) = (-x^0,x^1,x^2,x^3)\,, \\ &\Theta U(g_{x,y}) = U^\dagger(g_{\Theta x,\Theta y})\,, \\ &\Theta\psi(x) = i\gamma_0^E i\gamma_5^E \psi^*(\Theta x) \Rightarrow \Theta[\psi^\dagger(x) i\gamma_5^E] = -i\psi^T(\Theta x)\gamma_0^E \end{aligned} \tag{24}$$

and

$$\Theta(f(U)[\psi^\dagger(x_1)i\gamma_5^E]\cdots\psi(x_n)) = f^*(\Theta U)[-i\psi^T(\Theta x_1)\gamma_0^E]\cdots[i\gamma_0^E i\gamma_5^E\psi^*(\Theta x_n)]\,. \tag{25}$$

It is easy to verify that the proof of the OS positivity of the action (23) with the OS reflection operator as defined above is the same as that given in Ref. 5, with the independent fermions ψ_1 and ψ_2 used there, replaced by $\psi^\dagger(i\gamma_5^E)$ and ψ, respectively. The general proof of the OS positivity (for all separations) of the Wilson action obtained using our continuation procedure is presented elsewhere.[7]

The Hermiticity of the Euclidean action (23) may be useful in overcoming the problem of complex fermion determinant in the Monte Carlo simulation of high-temperature QCD as follows. Consider, for simplicity, the Euclidean partition function (20) without sources:

$$\begin{aligned} W^E &= \int \mathcal{D}(\psi^\dagger i\gamma_5^E)\mathcal{D}\psi \exp(S^E) \\ &= \mathrm{Det}(i\gamma_5^E)\mathrm{Det}[i\gamma_5^E(i\partial\!\!\!/^E - m)]\,. \end{aligned} \tag{26}$$

Here S^E is as given by (19). Notice that both the determinants are of Hermitian matrices. A similar procedure can be followed for a high-temperature lattice QCD to yield the partition function for nonzero chemical potential to be the determinant of a Hermitian matrix. This is in contrast to the partition function obtained using the usual continuation procedure, which is equal to the determinant of a non-Hermitian matrix and hence difficult to simulate numerically. A similar problem arises in the Monte Carlo simulation of the chiral Schwinger model. The implications of the new continuation procedure for this problem are discussed elsewhere.[6]

(IV) Conclusions and discussion.—We have found a one-complex-parameter family of Dirac actions which is invariant under the interpolating Lorentz transformations. When the interpolating parameter is equal to 0, this action reduces to the usual Minkowski-space Dirac action, and when equal to $\pi/2$, the action corresponds to an SO(4)-invariant Hermitian action. Thus, a procedure has been developed to go continuously from Minkowski space to Euclidean space (and vice versa). In particular, there is no doubling of degrees of freedom of the Dirac fermion. The Euclidean action so obtained leads to the correct Euclidean propagator and, contrary to the claim in Ref. 1, there is no contradiction between the SO(4) covariance and the form of the propagator. Further, it has been argued that the lattice action obtained using the present method of continuation satisfies the OS positivity condition.

Some of the interesting features of the continuation procedure presented here are as follows. The Euclidean Wilson action is Hermitian and the partition function for the high-temperature QCD can be expressed as the determinant of a Hermitian operator. The Euclidean chiral transformation is a chiral scale transformation (rather than chiral phase transformation), and this leads to a better understanding of the chiral anomalies.[6]

Further, it will be shown elsewhere[3] that, when any supersymmetric theory is continued using the present procedure, the resultant Euclidean theory is supersymmetric.

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[a]Electronic address: physics%vigyan.ernet.in@uunet.uu.net.shakti.vigyan.ernet.in.

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