# Euclideanization of Majorana and Weyl Fermions

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**Abstract**

A *continuous* procedure is presented for Euclideanization of Majorana and Weyl fermions *without* doubling their degrees of freedom. The Euclidean theory so obtained is SO(4) invariant and OS positive. This enables us to define a one complex parameter family of the N=1 supersymmetric Yang-Mills (SSYM) theories which interpolate between the Minkowski and a Euclidean SSYM theory. The interpolating action, and hence the Euclidean action, manifests all the continuous symmetries of the original Minkowski space theory.

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It is known that there are no Majorana fermions in four dimensional Euclidean space. Also, it is not possible to have an SO(4) invariant action for a single Weyl fermion. Hence it is believed that when Minkowski space theories with Majorana or Weyl fermions are Euclideanized the fermionic degrees of freedom have to be doubled.

In fact even when a Dirac fermion is Euclideanized using the usual procedure[1] its degrees of freedom are doubled. The Euclidean Majorana or Weyl fermion is obtained by imposing appropriate constraints[2,3] on the Euclidean Dirac fermion. Thus the degrees of freedom of a Majorana or a Weyl fermion too are doubled when they are Euclideanized using the usual procedure.

In an earlier paper[4] we presented a new procedure for Euclideanizing Dirac fermions. Unlike the usual procedure the new procedure was continuous and the degrees of freedom of the Dirac fermion were not doubled. However, due to the difficulty with SO(4) invariance, as discussed above, the extension of the new continuation procedure to the case of Majorana and Weyl fermions is not obvious.

In this paper we give a *continuous* procedure for Euclideanization of Weyl and Majorana fermions *without* doubling their degrees of freedom. This is achieved by defining one complex parameter family of "Majorana like" and Weyl fermions. The Euclidean theories so obtained are SO(4) invariant, Osterwalder-Schrader (OS) positive and manifest all the continuous symmetries of the original Minkowski space theory. This allows us to define one complex parameter family of N=1 supersymmetric Yang-Mills (SSYM) theories which interpolate between the Minkowski space N=1 SSYM and Euclidean SSYM theories.

In Section 1 we discuss an apparent non-uniqueness in the new continuation procedure for Euclideanization of the Dirac fermions. One complex parameter families of Weyl and Majorana fermions are defined in Sections 2 and 3. A family of N=1 SSYM theories are discussed in Section 4.

## 1. $\sqrt{-g^\theta}$.

The Dirac action was continued to Euclidean space using an interpolating action[4]

$$S^\theta = \int d^4x \sqrt{-g^\theta}\ \psi^\dagger M\ [i(g^\theta)^{\mu\nu}\gamma^\theta_\mu \partial_\nu - m]\psi. \tag{1}$$

The notation is as in ref. [4]. In particular the interpolating metric and gamma matrices are defined in terms of the complex interpolation parameter $\theta$ as

$$g^\theta_{\mu\nu} = (\frac{\cos 2\theta}{|\cos 2\theta|}, -1, -1, -1), \quad (0 \le \theta \le \pi/2,\ \theta \neq \pi/4)$$

$$\gamma^\theta_0 = \frac{1}{\sqrt{|\cos 2\theta|}}(\gamma_0 \cos\theta + i\gamma_5 \sin\theta), \quad \gamma^\theta_i = \gamma_i, \quad M = \gamma_0,$$

$$\gamma^\theta_5 = \frac{1}{\sqrt{|\cos 2\theta|}}(\gamma_5 \cos\theta - i\gamma_0 \sin\theta), \quad \Sigma^\theta_{\mu\nu} = \frac{1}{4}[\gamma^\theta_\mu, \gamma^\theta_\nu]. \tag{2}$$

(Here $M = \gamma_0, \gamma_0^\dagger = +\gamma_0$ and $\gamma_i^\dagger = -\gamma_i$.) These gamma matrices satisfy the algebra

$$\{\gamma^\theta_\mu, \gamma^\theta_\nu\} = 2g^\theta_{\mu\nu} \quad \text{and} \quad \{\gamma^\theta_\mu, \gamma^\theta_5\} = 0.$$

When $\theta = 0$ the action (1) reduces to the usual Minkowski space Dirac action and at $\theta = \pi/2$ it reduces to an SO(4) invariant Euclidean action[4].

When the vacuum amplitude is continued to Euclidean space, $\sqrt{-g^\theta}$ in (1) produces the required factor of '$i$' to go from the Feynman measure to the Wiener measure. However, only in curved space the volume element $d^4x$ has to be multiplied by $\sqrt{-g}$ so as to make it invariant under the general coordinate transformations. Since we are working in flat spacetime there is no symmetry argument which requires such a factor. In fact we are free to multiply various terms in the action by arbitrary powers of $\sqrt{-g^\theta}$. Since $\sqrt{-g^\theta} = 1$ for $\theta = 0$, various interpolating actions so obtained will give the same Minkowski space action but different Euclidean actions. This apparent arbitrariness in Euclideanization can be avoided if we impose the following (natural) requirements on the continuation procedure :

(a) The Euclideanization procedure should be continuous.

(b) The interpolating action should manifest *all* the continuous symmetries of the original Minkowski space theory.

(c) The Euclidean theory obtained from a unitary Minkowski space theory should satisfy the OS conditions, in particular the OS positivity condition.

Thus it is only with the volume element of the form used in (1) that the Euclidean partition function, and hence the Euclidean Greens functions, can satisfy the OS positivity condition. Insertion of powers of $g^\theta$ in different terms in the interpolating action will either lead to a trivial redefinition of the Euclidean gamma matrices (by a minus sign) or will violate (b) or (c). Thus the interpolating action (1) is "unique" and there is no ambiguity in Euclideanization of the Dirac Fermion.

In fact, as we will see presently, judicious insertion of $\sqrt{-g^\theta}$ enables us to continue Weyl and Majorana fermions to Euclidean space without doubling their degrees of freedom.

## 2. Euclideanization of Weyl fermions.

Using the definitions of the interpolating gamma matrices (2) we define the following orthogonal, positive and negative helicity projection operators for *all* the allowed values of $\theta$:

$$P_\pm = \frac{(1 \pm (-g^\theta)^{-1/2})\gamma_5^\theta)}{2}. \tag{3}$$

Note that without the factor $(-g^\theta)^{-1/2}$ these will not be projection operators for all the values of $\theta$. Clearly, for $\theta = 0$ these reduce to the Minkowski space projection operators. The invariant interpolating action $S_\pm^\theta$ for Weyl fermions with positive or negative helicity is then defined in terms of the positive or negative helicity projections $P_\pm\psi$ of a Dirac fermion $\psi$ as :

$$S_\pm^\theta = \int d^4x \sqrt{-g^\theta}\ (\psi^\dagger M P_\mp)\ i(g^\theta)^{\mu\nu}\gamma_\mu^\theta \partial_\nu (P_\pm\psi). \tag{4}$$

Since the projection operators commute with $\Sigma_{\mu\nu}^\theta$, the invariance of this action under the interpolating Lorentz transformations[4] is obvious. $P_\pm$ are projection operators for all the values of $\theta$ and the Minkowski space action involves only one Weyl fermion, i.e. only two (complex) fermionic degrees of freedom. Hence the interpolating action also depends on only two fermionic degrees of freedom. Thus there is no doubling of Weyl fermions during Euclideanization. Further, $(P_+\psi)$ and $(\psi^\dagger M P_-)$ transform according to the (0,1/2) and (1/2,0) representation of the "interpolating Lorentz group" (defined in ref. [4]) respectively. They are related by hermitean conjugation only when $\theta = 0$. For $\theta = \pi/2$ this action leads to the following Euclidean action for Weyl fermions:

$$S^E_\pm = \int d^4x \; (\psi^\dagger i\gamma_5^E P_\mp) i \; \partial\!\!\!/^E (P_\pm \psi) = \mp \int d^4x \; \psi^\dagger i \; \partial\!\!\!/^E P_\pm \psi. \qquad (5)$$

All the Euclidean gamma matrices are antihermitean and $P_\pm = (1 \pm i\gamma_5^E)/2$ are the Euclidean helicity projection operators.

The exponent of the effective action for the Weyl fermions for any value of $\theta$ is defined as

$$Z^\theta_\pm = \int \mathcal{D}(\psi^\dagger M P_\mp) \; \mathcal{D}(P_\pm\psi) \exp(iS^\theta)$$

Which, by standard arguments, equals $\sqrt{Det \; \partial\!\!\!/^E}$ in Euclidean space. Introducing the source terms $(\xi^\dagger M P_\mp)(P_\pm\psi)$ and $(\psi^\dagger M P_\mp)(P_\pm\xi)$ the functional differentiation of the resulting partition function leads to Greens functions for all the values of $\theta$ for a Weyl fermion. These reduce to the usual Minkowski and Euclidean Greens funcions for $\theta = 0$ and $\pi/2$ respectively. Using a procedure similar to that of ref. [4] it is easy to verify that the above partition function is also OS positive.

Thus, unlike the usual procedure of Euclideanizing Weyl fermion, we have gone continuously from the Minkowski space action to a Euclidean action and there is no doubling of the fermionic degrees of freedom.

## 3. Euclideanization of Majorana Fermions.

For arbitrary values of the interpolating parameter $\theta$, the charge conjugate of a Dirac fermion $\psi$ is given by[4]

$$i\psi^c = \mathcal{C}\psi = i\gamma_2\gamma_0^\theta(\overline{\psi})^T = iC\overline{\psi}^T; \; C^{-1}(\gamma_\mu^\theta)^T C = -\gamma_\mu^\theta, C^2 = -g^\theta. \qquad (6)$$

Where $\overline{\psi} = \psi^\dagger\gamma_0$ and we have chosen a basis for the gamma matrices such that $\gamma_2$ is antisymmetric and all the other gamma matrices are symmetric. Further, $C^{-1} = (-g^\theta)^{-1}\gamma_2\gamma_0^\theta$. We now define a pair of fermions

$$\psi_M = \psi + \frac{\psi^c}{\sqrt{-g^\theta}}, \quad \overline{\psi}_M = \overline{\psi} + \frac{\overline{\psi}^c}{\sqrt{-g^\theta}}. \tag{7}$$

(Here $\overline{\psi}^c = (\psi^c)^\dagger\gamma_0$ and the $\dagger$ does not act on $\theta$; $\overline{\psi}_M = \psi_M^\dagger\gamma_0$ only for $\theta = 0$.) related by the Majorana condition

$$\psi_M = \frac{C}{\sqrt{-g^\theta}}(\overline{\psi}_M)^T, \tag{8}$$

for all the allowed values of $\theta$. For $\theta = 0$ eq.(6-8) reduce to the usual Minkowski space charge conjugation. Using (6) it it is easy to see that under the interpolating Lorentz transformations[4] $\psi^c$ transforms like $\psi$ and $\overline{\psi}_M$ transforms as the inverse of $\psi_M$. Thus the action for a free Majorana fermion for arbitrary values of $\theta$ is given by

$$S_M^\theta = \int d^4x\sqrt{-g^\theta}\ \overline{\psi}_M\ [i(g^\theta)^{\mu\nu}\gamma_\mu^\theta\partial_\nu - m]\psi_M. \tag{9}$$

It goes over to the usual action for Majorana fermion in Minkowski space for $\theta = 0$:

$$S_M^M = \int d^4x\ \overline{\psi}_M\ [i\ \partial\!\!\!/ - m]\psi_M. \tag{10}$$

Using (6) we see that the interpolating action (9) describes a *chargeless* fermion for all values of $\theta$. For $\theta = \pi/2$ the interpolating action reduces to the following Euclidean action for "Majorana like" fermions

$$S^{\pi/2} = iS_M^E = \int d^4x\ (\overline{\psi} - i\overline{\psi}^c)[i\ \partial\!\!\!/^E - m](\psi - i\psi^c). \tag{11}$$

Since we have only one Dirac fermion satisfying one constraint for all the values of $\theta$, there are only two fermionic degrees of freedom for all the values of $\theta$. Further, the Euclidean Majorana fermion is defined in terms of a Dirac fermion, hence the OS positivity of a Euclidean theory with such Majorana fermions can be shown using a procedure similar to that of ref. [4]. Thus we have continued the action for Majorana fermion to Euclidean space such that the Euclidean action is SO(4) invariant and the degrees of freedom of the Majorana fermion are not doubled. The pair of fermions satisfying the Majorana condition are related by hermitean conjugation only in Minkowski space. Following a procedure similar to that of the earlier section it is easy to see that the Euclidean action (11) leads to the correct Euclidean Greens functions for Majorans fermions.

## 4. One parameter family of N=1 SSYM theories.

Having obtained Majorana and Weyl fermions for arbitrary values of the interpolating parameter $\theta$, we can now define a one parameter family of N=1 SSYM theories as follows.

When the Weyl fermion in the interpolating action (4) is coupled to a non-abelian gauge field, both transforming according to the adjoint representation of some gauge group, we get (the gauge group indices are suppressed for convenience.)

$$S^{\theta}_{\pm} = \int d^4x \sqrt{-g^{\theta}}\ [(\psi^{\dagger} M P_{\mp})\ i(g^{\theta})^{\mu\nu}\gamma^{\theta}_{\mu} D_{\nu}(P_{\pm}\psi) - 1/4 F_{\mu\nu}F^{\mu\nu}], \tag{12}$$

which is invariant under the following SUSY transformations

$$\delta_{\epsilon} A_{\mu} = i(\epsilon^{\dagger} M P_{\mp}\gamma^{\theta}_{\mu} P_{\pm}\psi - \psi^{\dagger} M P_{\mp}\gamma^{\theta}_{\mu} P_{\pm}\epsilon),$$

$$\delta_{\epsilon}(P_{\pm}\psi^{a}) = \Sigma^{\theta}_{\mu\nu} F^{\mu\nu} P_{\pm}\epsilon. \tag{13}$$

The action is not real and the SUSY transformations are not unitary for all the values of $\theta \neq 0$. To avoid this difficulty we define a different hermitean conjugation operator $\mathcal{H}$ such that, for all $\theta, \mathcal{H}A = A^{\dagger}$ where $A$ is any complex matrix and $\mathcal{H}(P_{\pm}\psi) = \psi^{\dagger} M P_{\mp} M$. Clearly the action and the SUSY transformation of $A_{\mu}$ are invariant under $\mathcal{H}$. Also, $\mathcal{H}$ commutes with the interpolating Lorentz transformations. For $\theta = 0$ this definition yields the usual hermitean conjugation.

The SUSY invariance of the action (12) can be easily verified using the fact that the fermions are Weyl for all values of $\theta$ and therefore by Fierz rearrangement

$$f^{abc}[\epsilon^{\dagger} M P_{\mp}\gamma^{\theta}_{\mu} P_{\pm}\psi^{a}][(\psi^{b})^{\dagger} M P_{\mp} M \gamma^{\theta}_{\mu} P_{\pm}\psi^{c}] = 0. \tag{14}$$

We can define, for all the allowed values of $\theta$, an N=1 SSYM action involving Majorana fermions in a similar fashion by gauging the action (9) for free Majorana fermions to yield

$$S^\theta = \int d^4x\sqrt{-g^\theta}\ [\overline{\psi}_M\ i(g^\theta)^{\mu\nu}\gamma^\theta_\mu D_\nu\psi_M - 1/4F_{\mu\nu}F^{\mu\nu}]. \tag{15}$$

This action is invariant under the SUSY transformations

$$\delta_\epsilon A_\mu = i(\overline{\epsilon}_M\gamma^\theta_\mu\psi_M - \overline{\psi}_M\gamma^\theta_\mu\epsilon_M)$$

$$\delta_\epsilon\psi_M = \Sigma^\theta_{\mu\nu}F^{\mu\nu}\epsilon_M. \tag{16}$$

Here $\epsilon_M$ and $\overline{\epsilon}_M$ are defined in the same way as $\psi_M$ and $\overline{\psi}_M$ in (7). Again the action (15) and the SUSY transformations (16) are not hermitean [3]. Hence we define hermitean conjugate of Majorana fermions as $\mathcal{H}\psi_M = \overline{\psi}_M M$, which coincides with the usual definition when $\theta = 0$. Using (6) the SUSY invariance of the above action can be easily verified. The SUSY algebra for arbitrary $\theta$ is similar to the usual Minkowski space N=1 SUSY algebra[5] with the Minkowski space Gamma matrices and the metric replaced by $\gamma^\theta_\mu$ and $g^\theta_{\mu\nu}$ respectively.

The actions (12) and (15) involve only one massless gauge field and only one Weyl or Majorana fermion. Further, they are invariant under the modified hermitean conjugation $\mathcal{H}$, hence we call them an N=1 SSYM. These actions lead to the usual N=1 SSYM action in Minkowski space when $\theta = 0$. For $\theta = \pi/2$ they lead to an N=1 SSYM in Euclidean space. Thus we have continued the N=1 SSYM to Euclidean space in a continuous way and without doubling the degrees of freedom of the Weyl or Majorana fermions. The interpolating theory and hence the Euclidean theory is supersymmetric. Further, it can be easily seen that the actions (12) and (15) are also invariant under chiral transformations[4]. Therefore, the interpolating theory, and hence the Euclidean theory, manifests *all* the continuous symmetries of the original Minkowski space theory. The argument for the OS positivity of these theories is the same as that given in Sections 2 and 3. Thus the continuation procedure satisfies all the requirements mentioned is Section 1.

## 5. Conclusions

We have argued that, while Euclideanizing a theory, one is free to insert arbitrary powers of $\sqrt{-g^\theta}$ in various terms in the interpolating action. The apparent non-uniqueness of the Euclidean action for Dirac fermions, so obtained, was removed by imposing some natural conditions of continuity and symmetry on the Euclideanization procedure. Further judicious insertions of powers of $\sqrt{-g^\theta}$ allowed us to define Weyl and Majorana fermions for all the values of the complex interpolation parameter $\theta$. Thus Majorana and Weyl fermions have been Euclideanized in a *continuous fashion without doubling* their degrees of freedom.

One parameter family of N=1 SSYM involving these Weyl or Majorana fermions were then defined. For $\theta = 0$ these led to the usual SSYM theory in Minkowski space and for $\theta = \pi/2$ they reduced to an OS positive N=1 SSYM in Euclidean space. Thus we have continued the N=1 SSYM to Euclidean space such that the interpolating theory, and hence the Euclidean theory, manifests all the symmetries of the original Minkowski space theory.

The Euclidean continuation of other supersymmetric theories, involving scalar fields, needs some more work and will be discussed elsewhere[6].

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