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ENERGY DENSITY OF BLACK BODY RADIATION AND QUANTIZATION

KNOWN ENERGY DENSITY OF BLACK BODY RADIATION

It was known at the end of the 19th century that the energy of black body radiation in a cavity was given by: of the form

$$u(\nu) = \frac{8\,\pi\,\nu^2}{c^3}\,F(\frac{\nu}{T}) \tag{1}$$

where ν is the frequency of the radiation, $\nu = \frac{c}{\lambda}$, λ is the wavelength. Defining $x = \frac{\nu}{T}$, we see that $F(x) \approx kT$ for $x << 1$. On the other hand, it was found that $F(x)$ decreases roughly like e^{-x} for $x >> 1$.

It can be seen from this that the total energy, obtained by integrating over frequencies is, apart from a constant factor:

$$U = \int_0^\infty u(\nu) d\nu = \int_0^\infty \nu^2\,T\,F(\frac{\nu}{T}) d\nu = const\,T^4 \int_0^\infty F(x) x^2 dx \tag{2}$$

The dependence on fourth power of the temperature constitutes Stefan's law.

CLASSICAL RESULT FOR ENERGY DENSITY OF BLACK BODY RADIATION

Consider a cubical box of length L. The electric field due to the radiation must be sinusoidal, subject to the boundary condition of vanishing at each of the edges. Then we can write:

$$E = sin(\frac{n_x\,\pi\,x}{L}) sin(\frac{n_y\,\pi\,y}{L}) sin(\frac{n_z\,\pi\,z}{L}) \tag{3}$$

The wavenumber of the radiation $k = \frac{2\pi\,\nu}{c}$ is given by:

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{(n_x^2\,+\,n_y^2\,+\,n_z^2)\,\pi^2}{L^2} = \frac{n^2\,\pi^2}{L^2} \tag{4}$$

where each of the n_i are positive integers.

The energy density of the radiation is equal to the number density of possible modes of frequency ν times the energy in each of the modes. The number density is given by 1 in terms of $n_x, n_y and n_z.$, or in terms of the total n,

$$N(n) dn = 2\,\frac{4\,\pi\,n^2}{8}\,dn = \pi\,n^2\,dn \tag{5}$$

The factor $4\,\pi\,n^2$ is just the surface area of a sphere of radius n, the factor 2 is present since we have two directions of polarization, and the factor $\frac{1}{8}$ appears since we only consider one octant of the sphere, with all three $n_i > 0$. Since $k = \frac{\pi\,n}{L}$, and $\nu = \frac{k\,c}{2\,\pi} = \frac{n\,c}{2\,L}$ we find:

$$N(\nu)d\nu = N(n)\frac{dn}{d\nu}d\nu = \pi\,n^2\frac{2\,L}{c}d\nu = \frac{8\,\pi\,\nu^2}{c^3}\,L^3 d\nu \tag{6}$$

Dividing by the volume, L^3 we get the number of modes per unit frequency. Now classically, each oscillation has an energy of kT. Thus the total energy density at frequency ν is given by:

$$u(\nu) = \frac{8\,\pi\,\nu^2}{c^3}\,kT \tag{7}$$

For low frequencies, this agrees with what was known experimentally. However, according to classical physics, the energy density would increase indefinitely with frequency. This would lead to infinite energy!

PLANCK'S BLACKBODY FORMULA

Planck proposed that the energy in each mode is quantized in multiples of a constant times the frequency. The constant is known as h. Thus the energy of oscillations is $E_\nu = n\,h\,\nu$, where n is an integer. Now from statistical mechanics we know that for a system in thermal equilibrium at temperature T, the probability of having energy E is a constant times $e^{-\frac{E}{kT}}$. Thus the probability of having energy $E_\nu = n\,h\,\nu$ is $e^{-\frac{n\,h\,\nu}{kT}}$, apart from normalization. Let us define a quantity $z = e^{-\frac{h\,\nu}{kT}}$. Then the average energy is $h\,\nu$ times the average number of quanta. The latter is given by:

$$N_{av} = \frac{z\,+\,2\,z^2\,+\,3\,z^3\,+...}{1\,+\,z\,+\,z^2\,+\,z^3\,+...} \tag{8}$$

In constructing these sums, Planck had to assume that the photons are indistiguishable. Using the binomial theorem, it is easy to show that the denominator is $\frac{1}{1-z}$, while the numerator $\frac{z}{(1-z)^2}$. We find for the average number of quanta

$$N_{av} = \frac{z}{1-z} = \frac{e^{-\frac{h\,\nu}{kT}}}{1-e^{-\frac{h\,\nu}{kT}}} = \frac{1}{e^{\frac{h\,\nu}{kT}}\,-1} \tag{9}$$

Multiplying by the number of modes of frequency ν, we obtain Planck's famous blackbody formula:

$$u(\nu) = \frac{8\,\pi\,\nu^2}{c^3}\,\frac{h\,\nu}{e^{\frac{h\,\nu}{kT}}\,-\,1} \tag{10}$$