Fractal Geometry of Minimal Spanning Trees

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Part I: arXiv:0902.3651
Part II: arXiv:0909.5343
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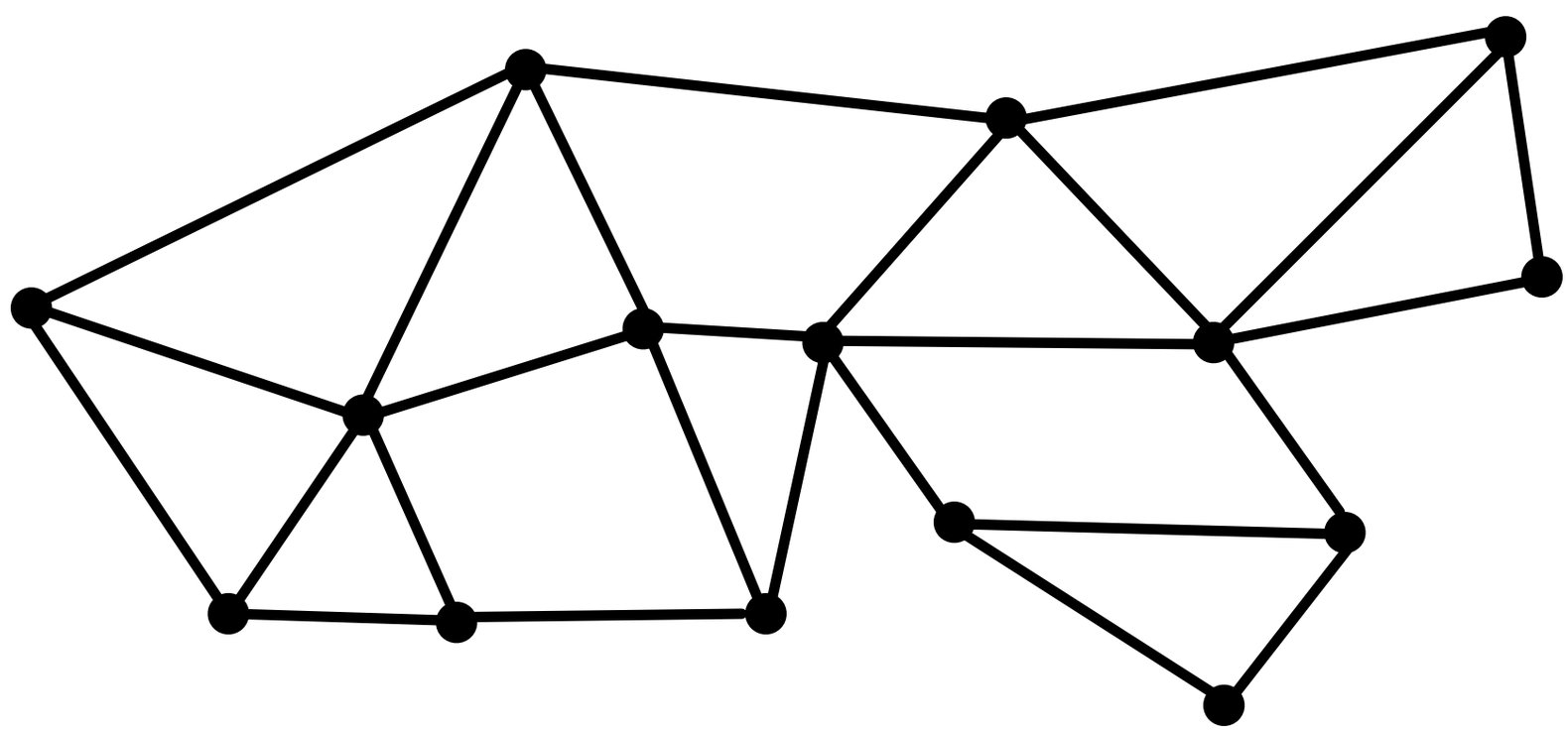
Problem Definition

- Given a graph with costs on the edges, the MST is
  - A tree (no loops)
  - Spanning: all vertices on the tree
  - Minimum: the spanning tree which minimizes total cost of edges on the tree (= *energy*)



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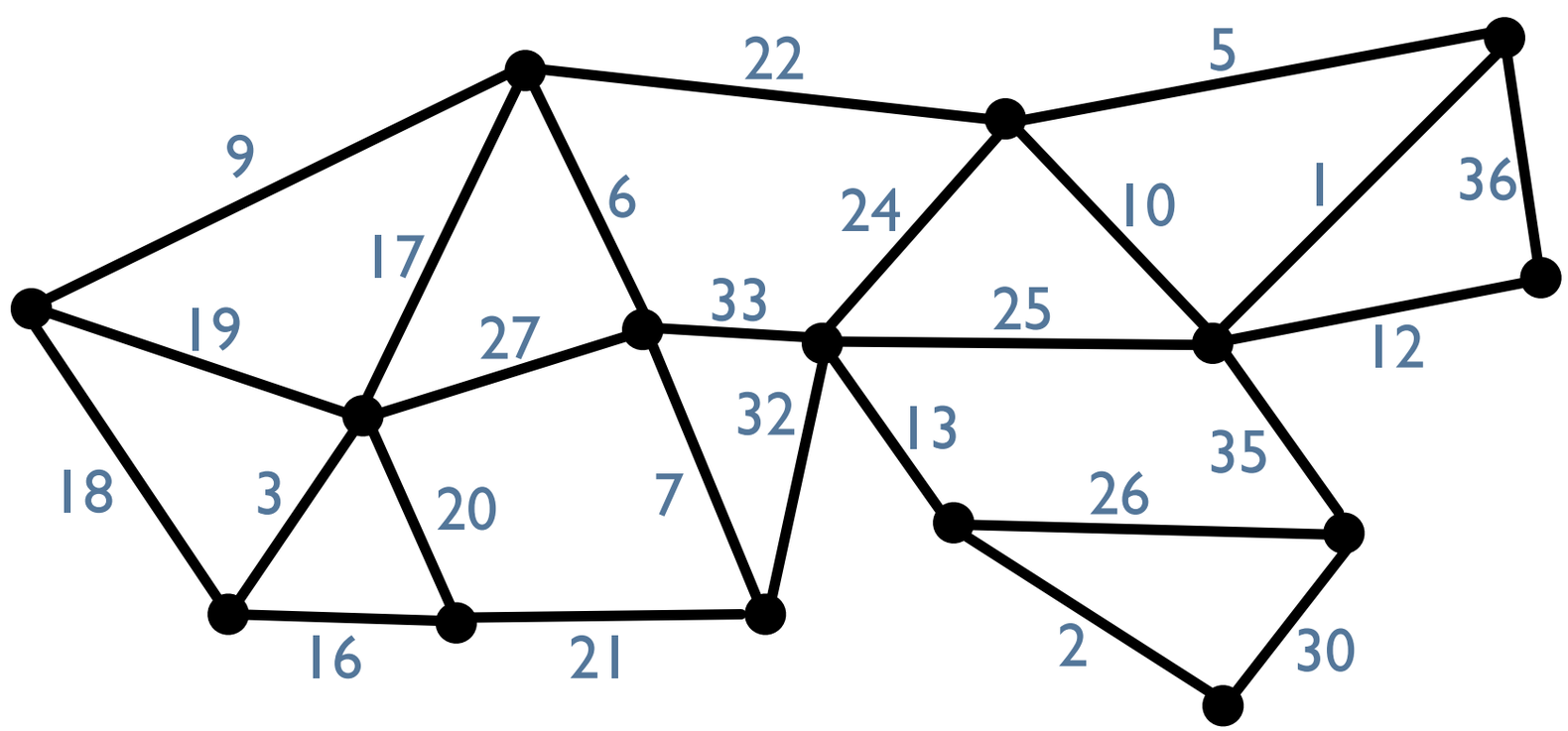

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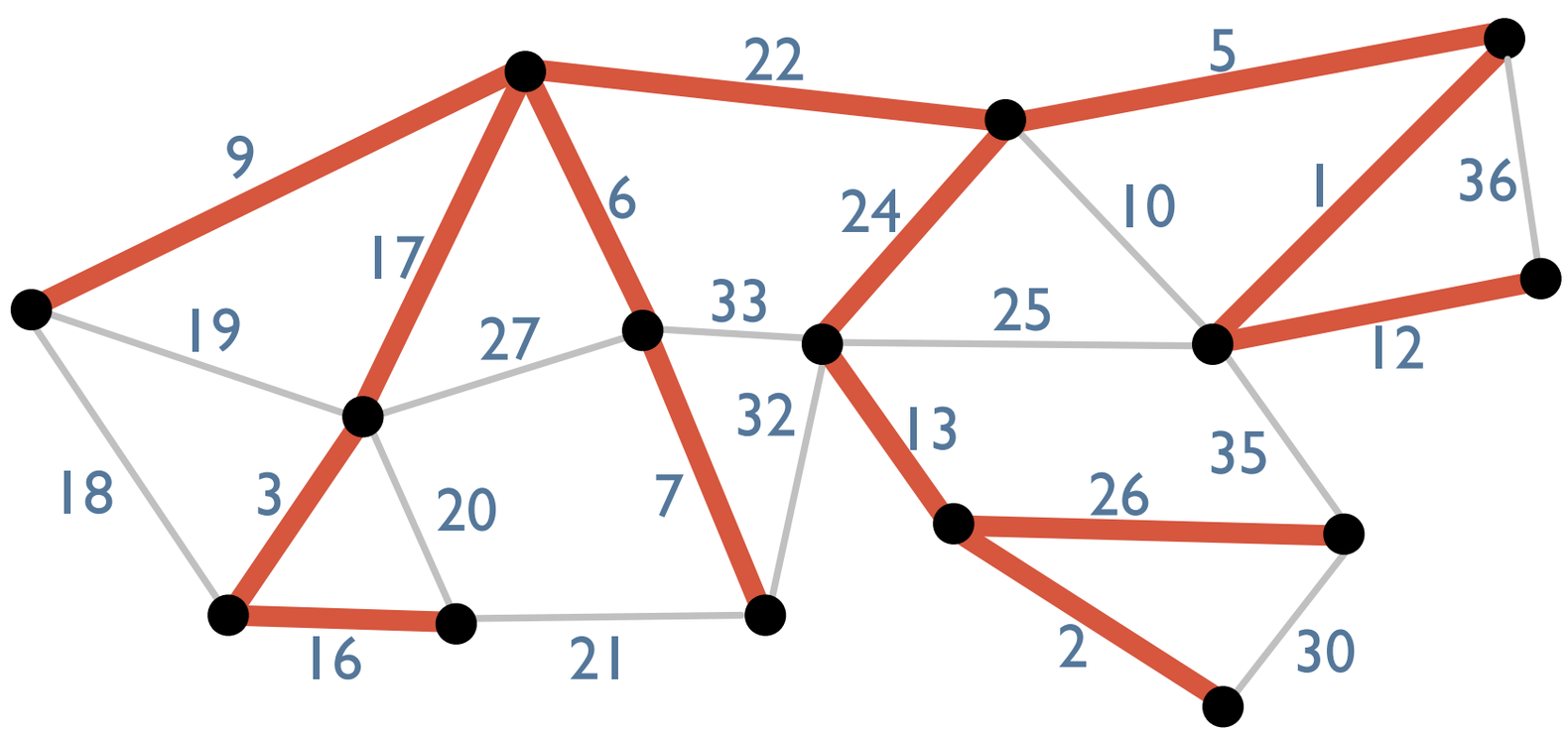
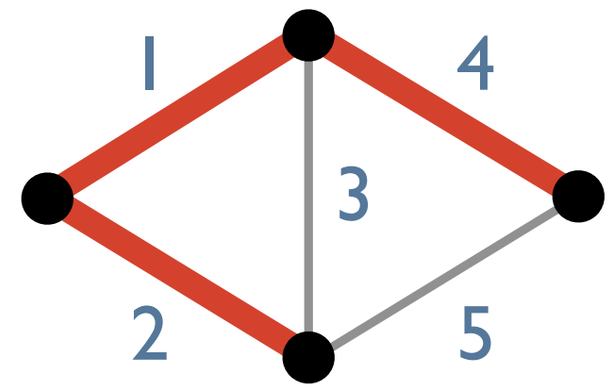
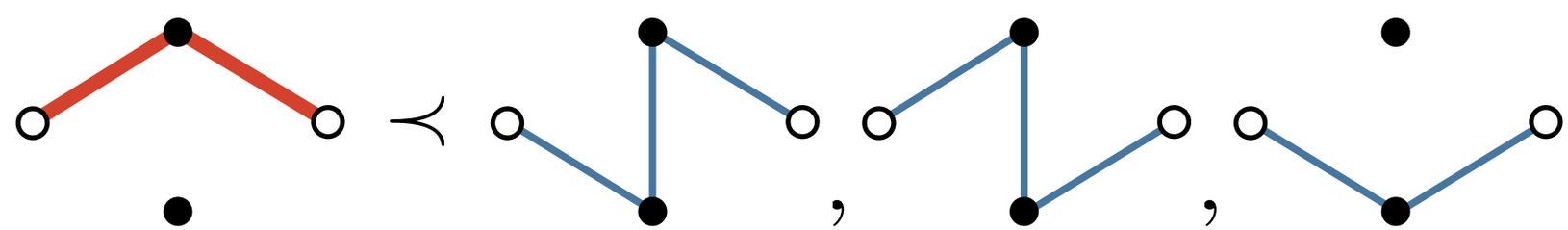


Relevant properties

Invariance under
reparameterization of costs



MST = Union of max. cost edge (*barrier*)-minimizing paths (*geodesics*)

e.g., 

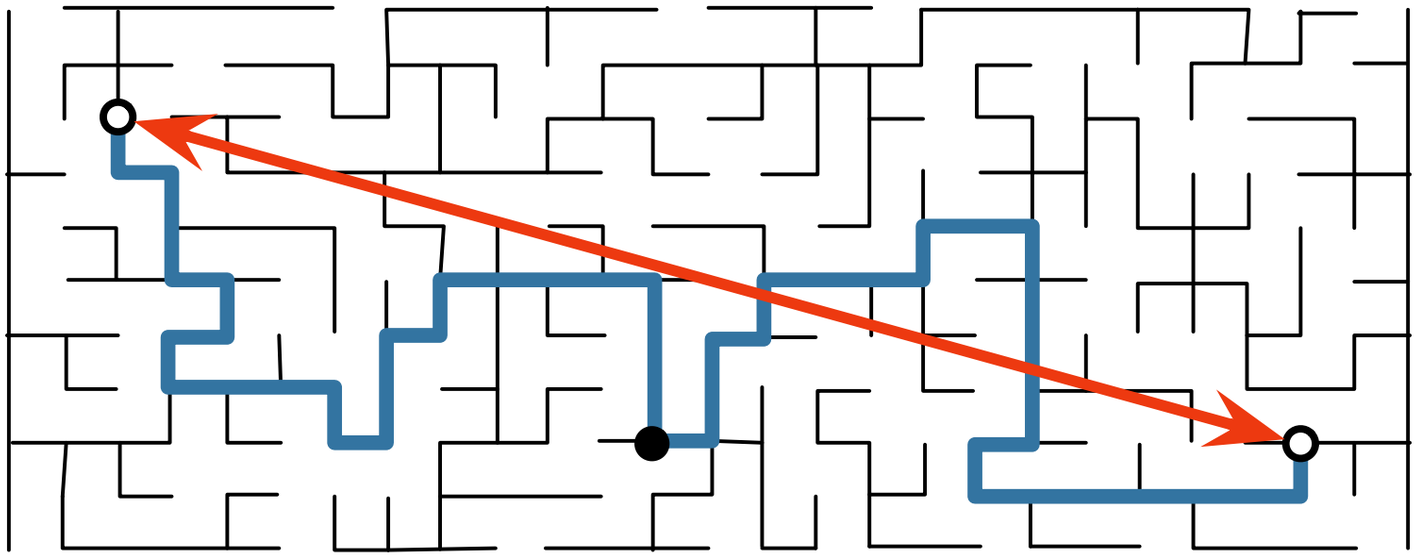
Construction via *greedy* algorithms
(no metastability, no RSB)

Motivation

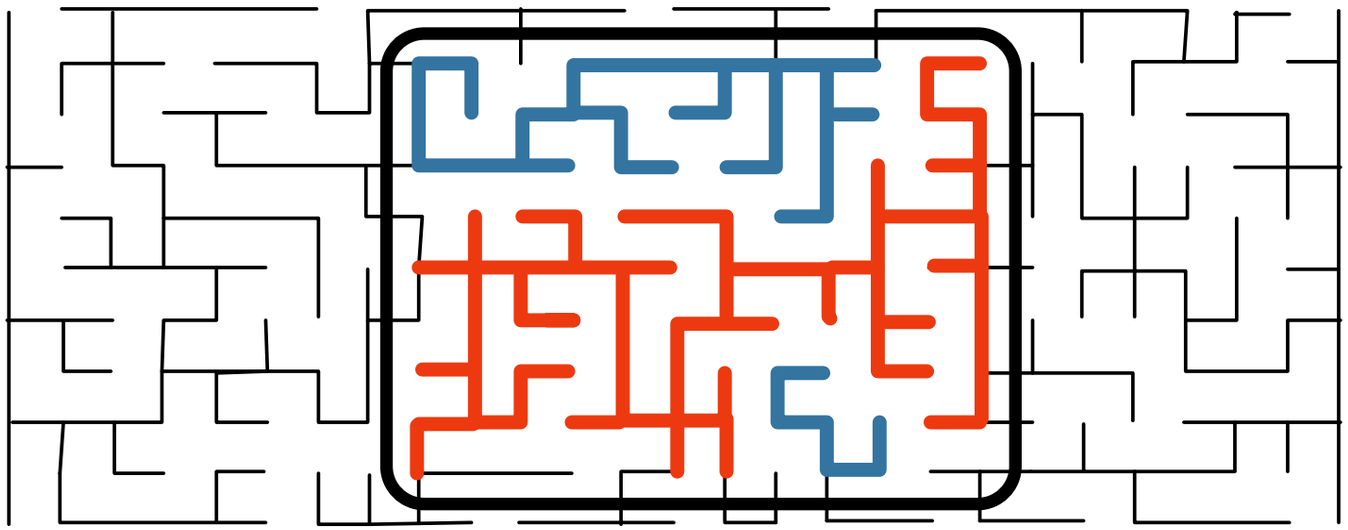
- MST paths = solutions to strong disorder transport problems
  - "Strong disorder" = extensive variance
  - E.g., resistor networks, polymers
- Nontrivial results (exponents) for a model with quenched disorder
  - Equivalently: nonlocal optimization problem in terms of local field theory
  - Newman and Stein '94: mapped EA spin glass ground state in strong disorder limit to MSTs

Objects of study

D of MST paths via *path vertex* connectedness function



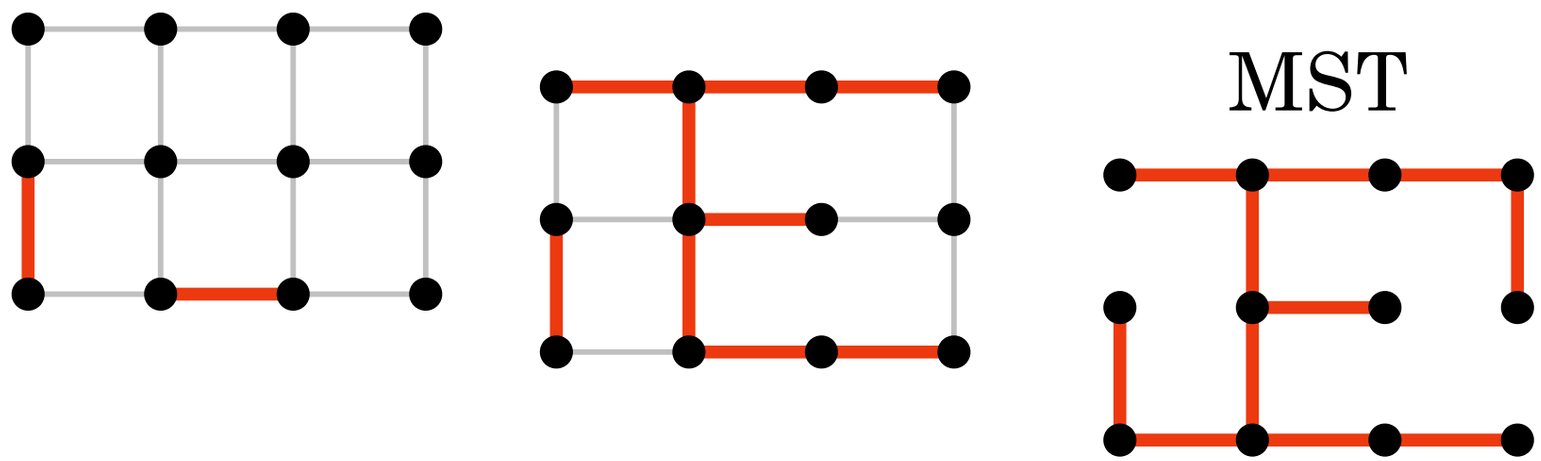
D of MSF clusters visible in window
$\rightarrow d_c$, via hyperscaling breakdown
($\rightarrow$ SD spin glass ground state enumeration)

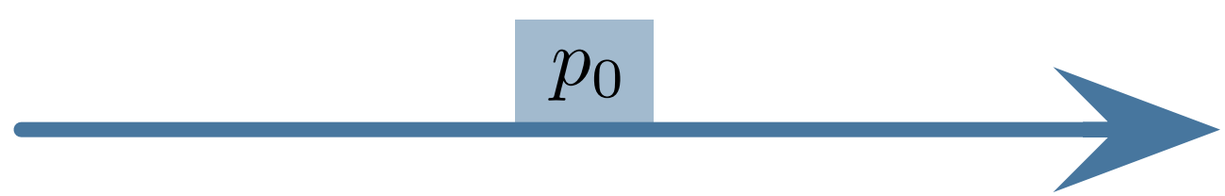


Connection to Percolation

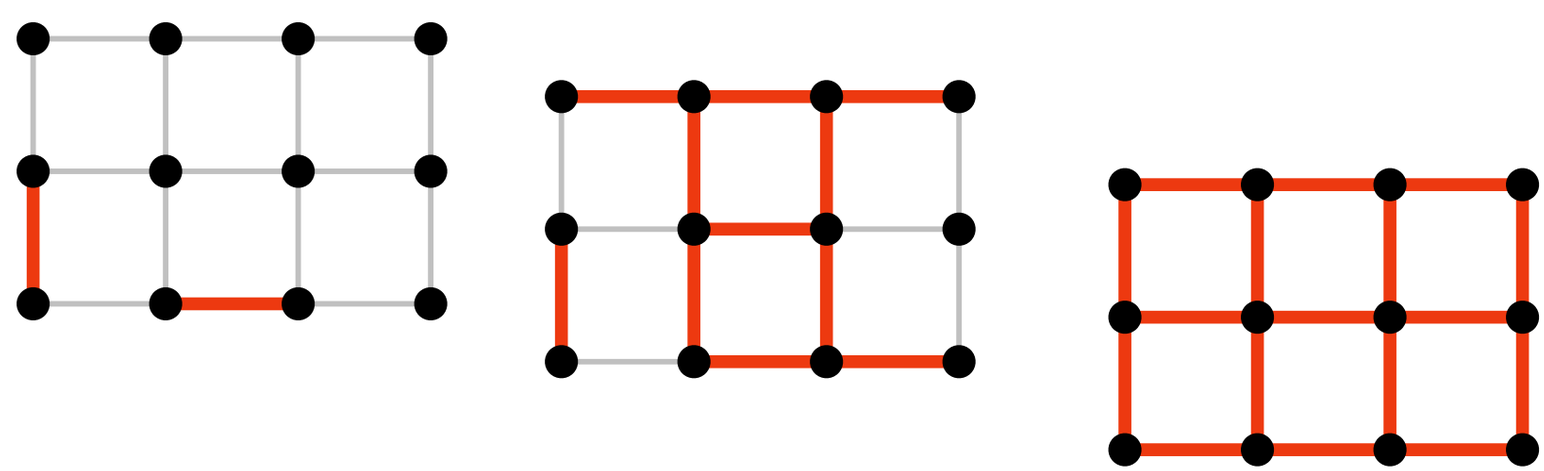
Disorder: edge costs are IID quenched random variables uniformly distributed on [0,1]

Kruskal's *greedy* algorithm for MST: add cheapest remaining edge unless a loop forms





Bond percolation: raise occupation probability, always accept cheapest remaining edge



Rest of Talk

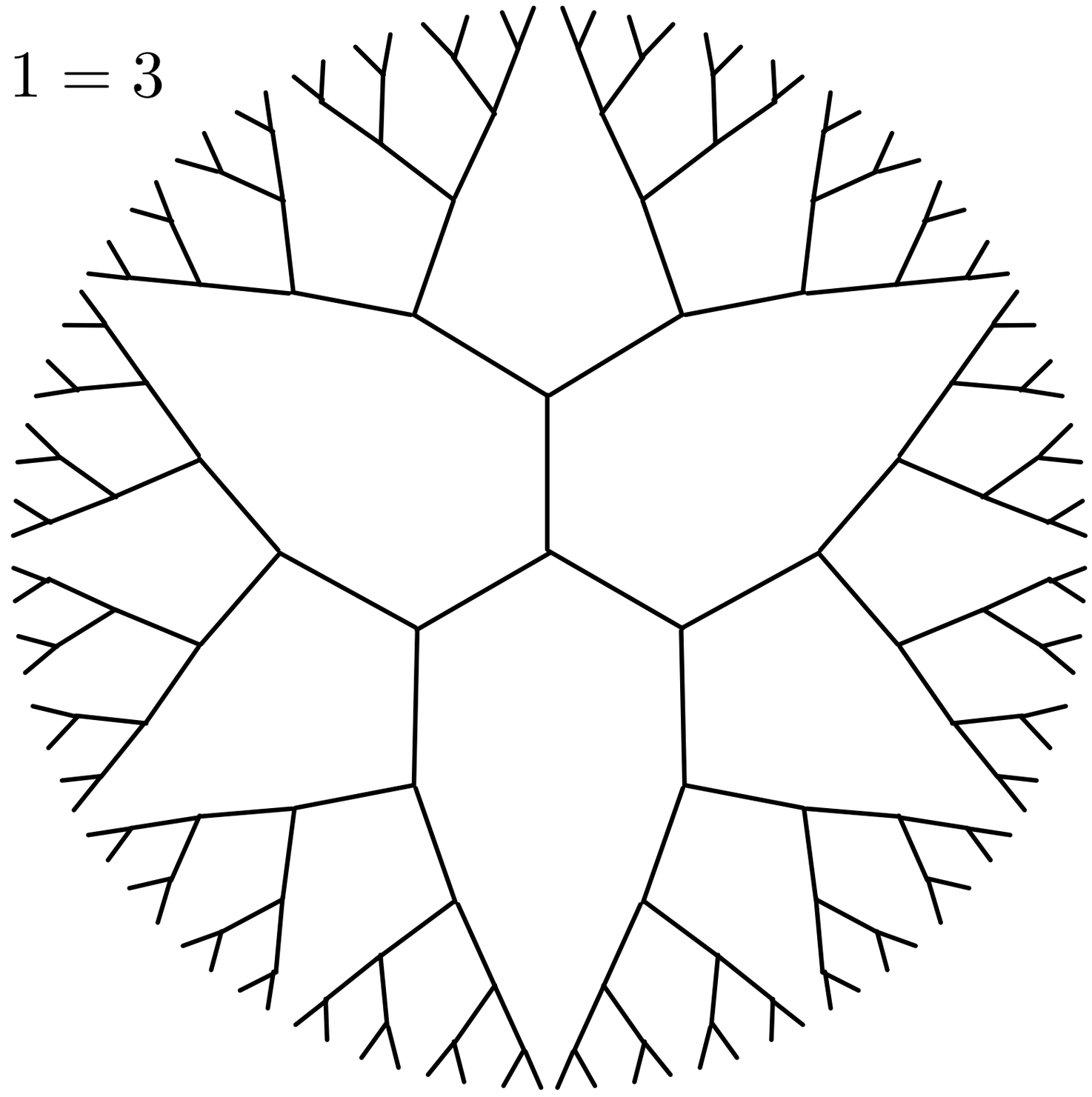
- Part I: Mean field theory on Bethe lattice
  - MFT breaks down at $d_c = 6$
  - How? MSF cluster proliferation from multiple percolation spanning clusters (e.g., Aizenman '97)
- Part II: Loop corrections for $d < d_c$
  - MSF cluster $D = d$, so look at MST paths
  - Adapt field theoretic perturbation series for percolation
  - Prove series is renormalizable to *all orders*
  - 1-loop example RG calculation

Part I: Mean field theory

No finite cycles

Bethe Lattice percolation

Coordination $z = \sigma + 1 = 3$

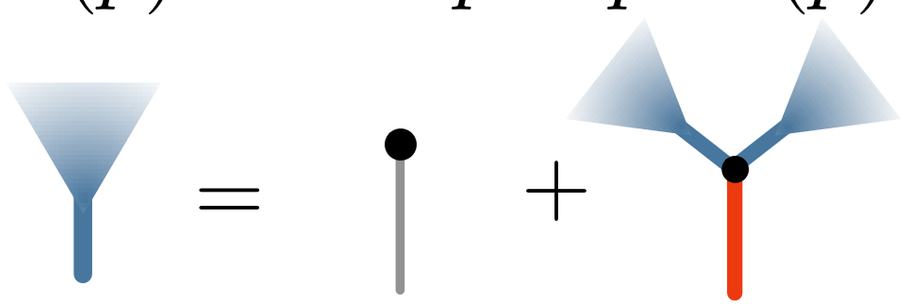


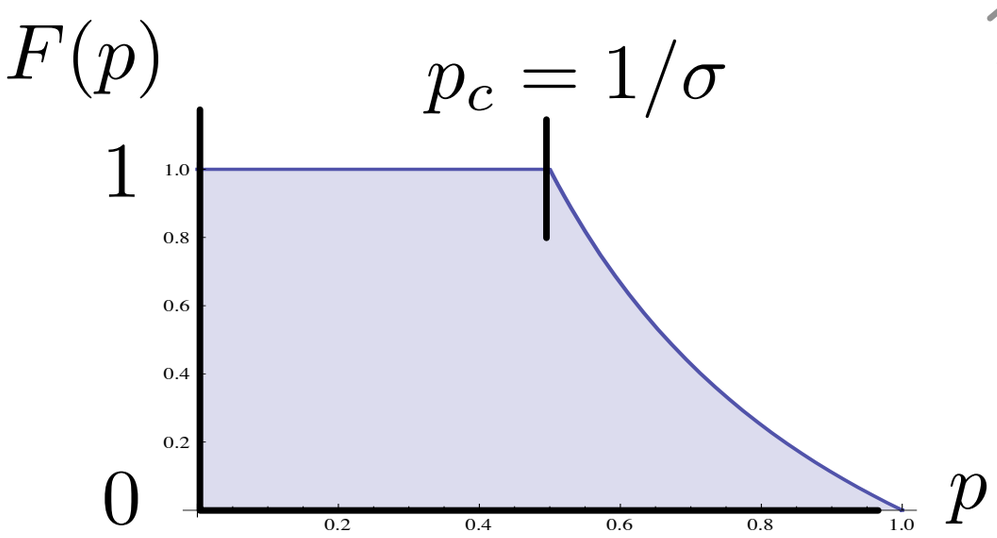
Bethe Lattice percolation

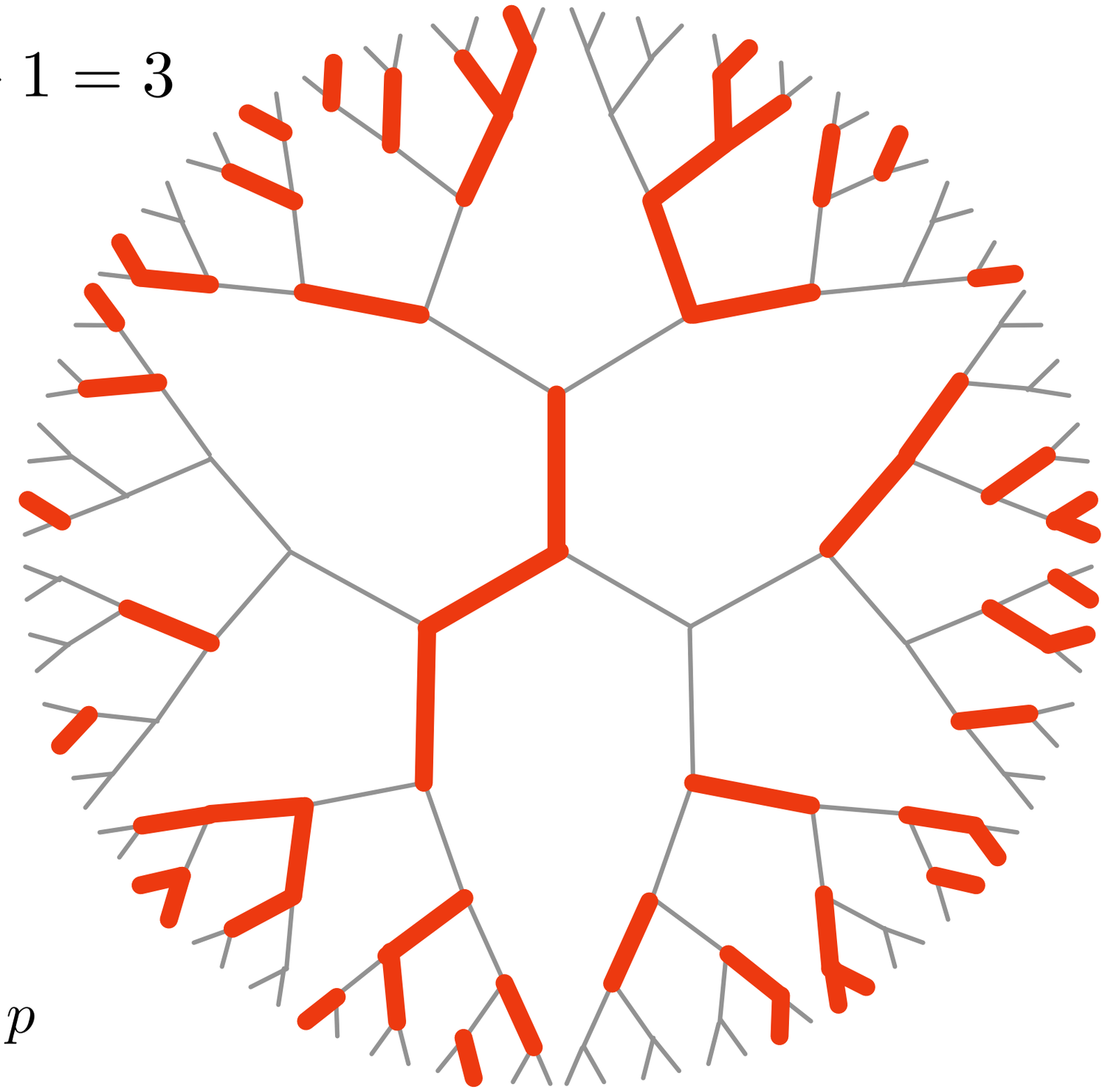
Coordination $z = \sigma + 1 = 3$

Percolation:

$$F(p) = 1 - p + pF^{\sigma}(p)$$

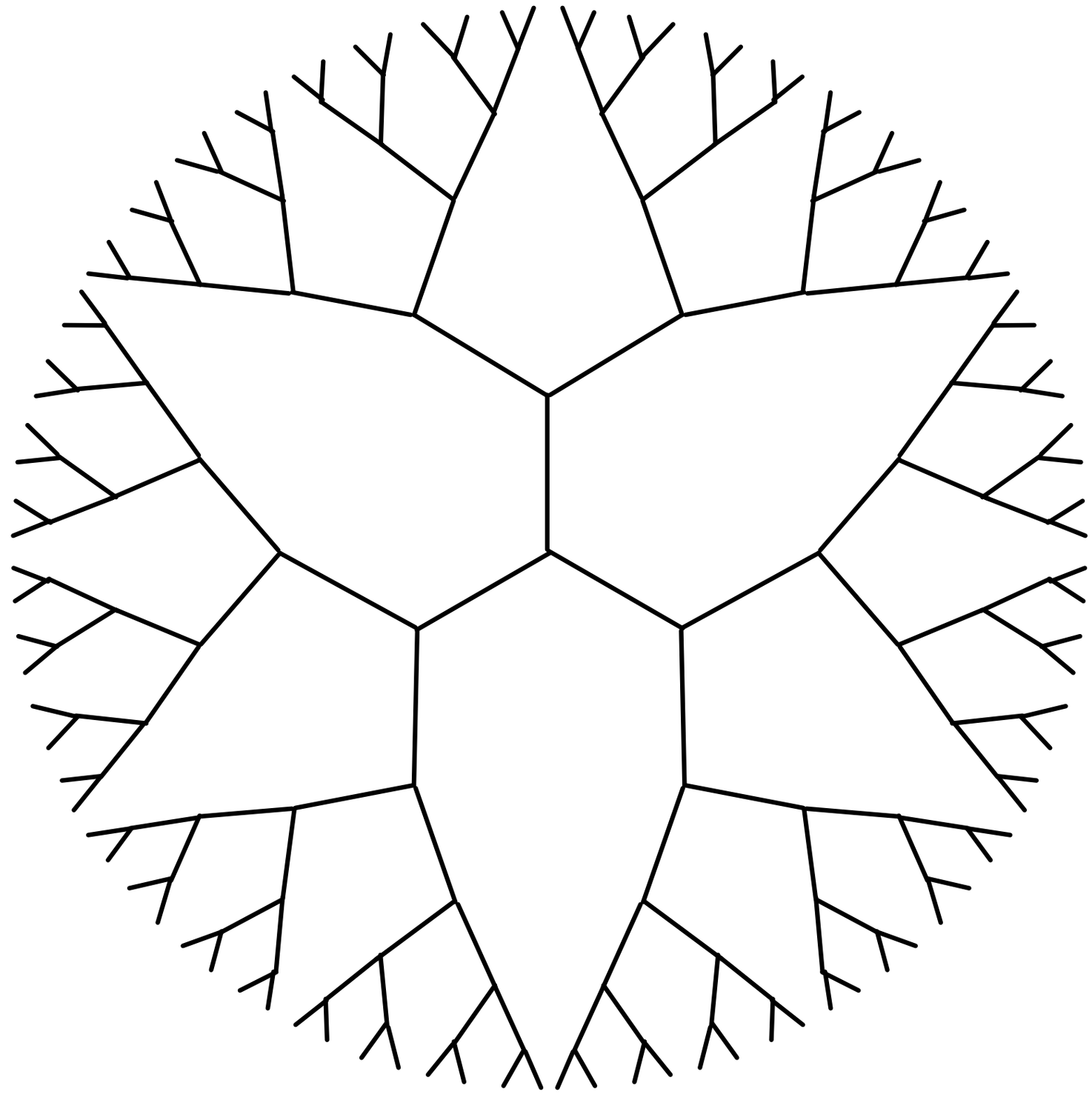






Bethe Lattice MSTs

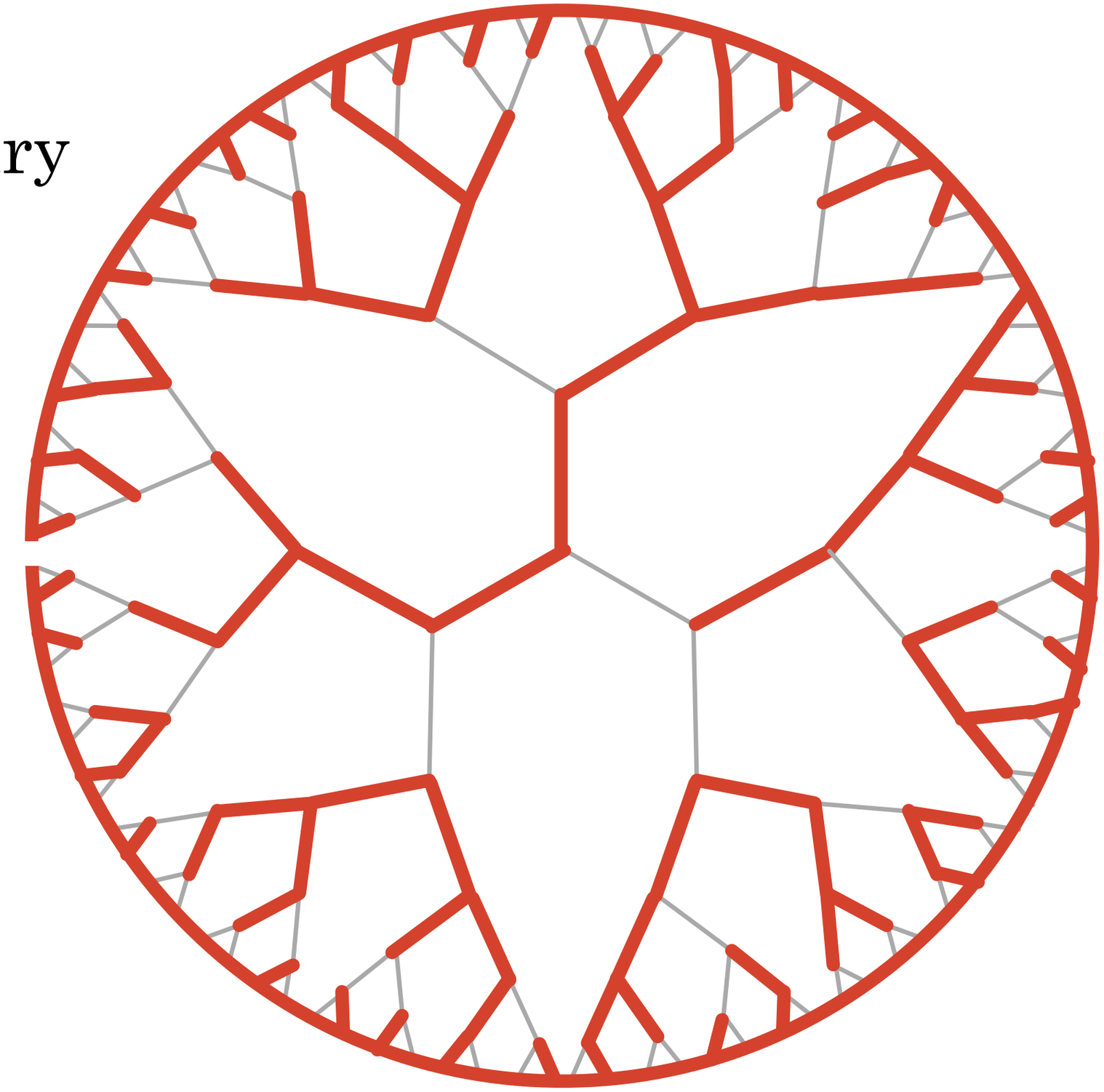
MST:



Bethe Lattice MSTs

MST:

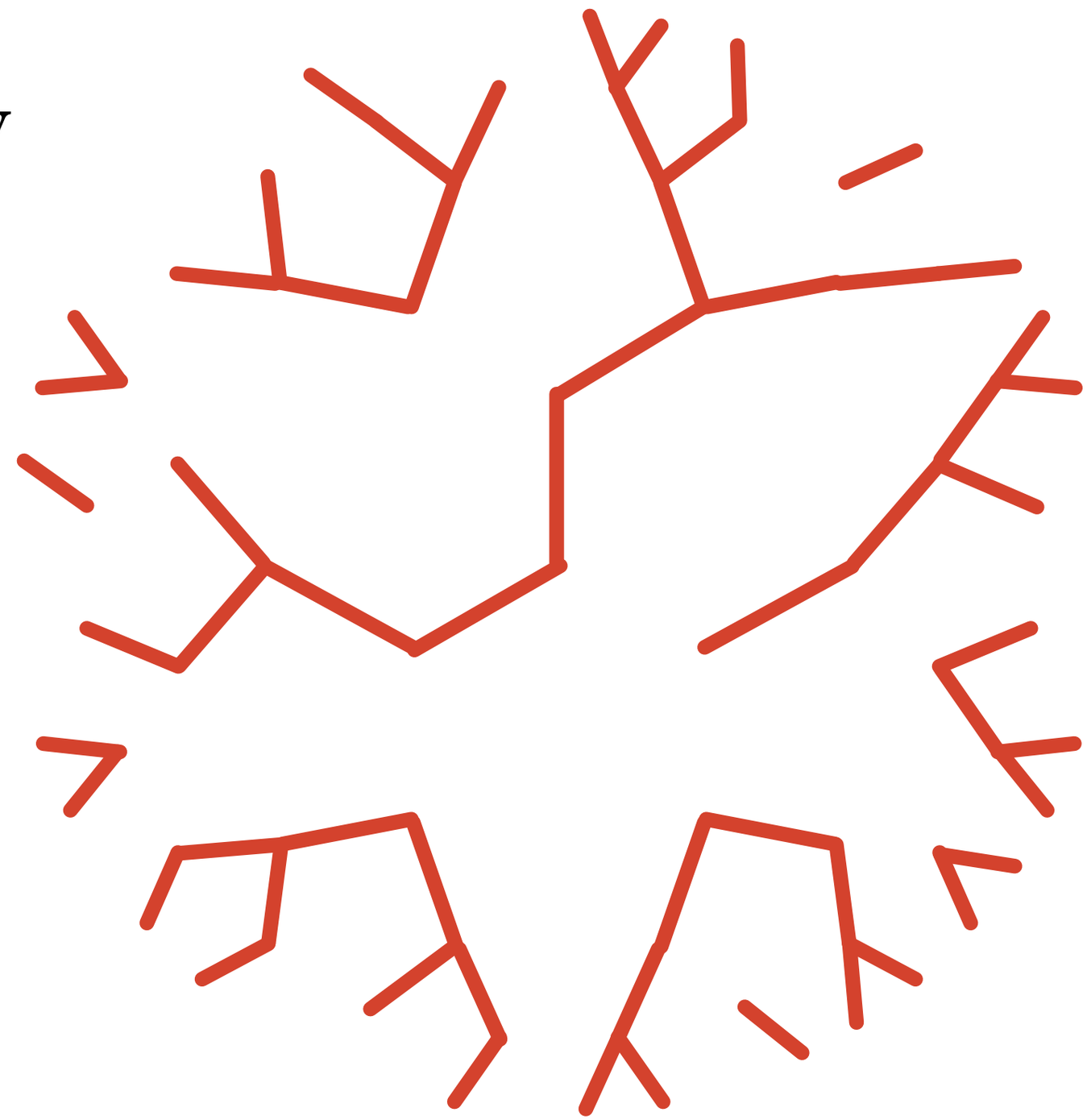
- Use wired boundary conditions



Bethe Lattice MSTs

MST:

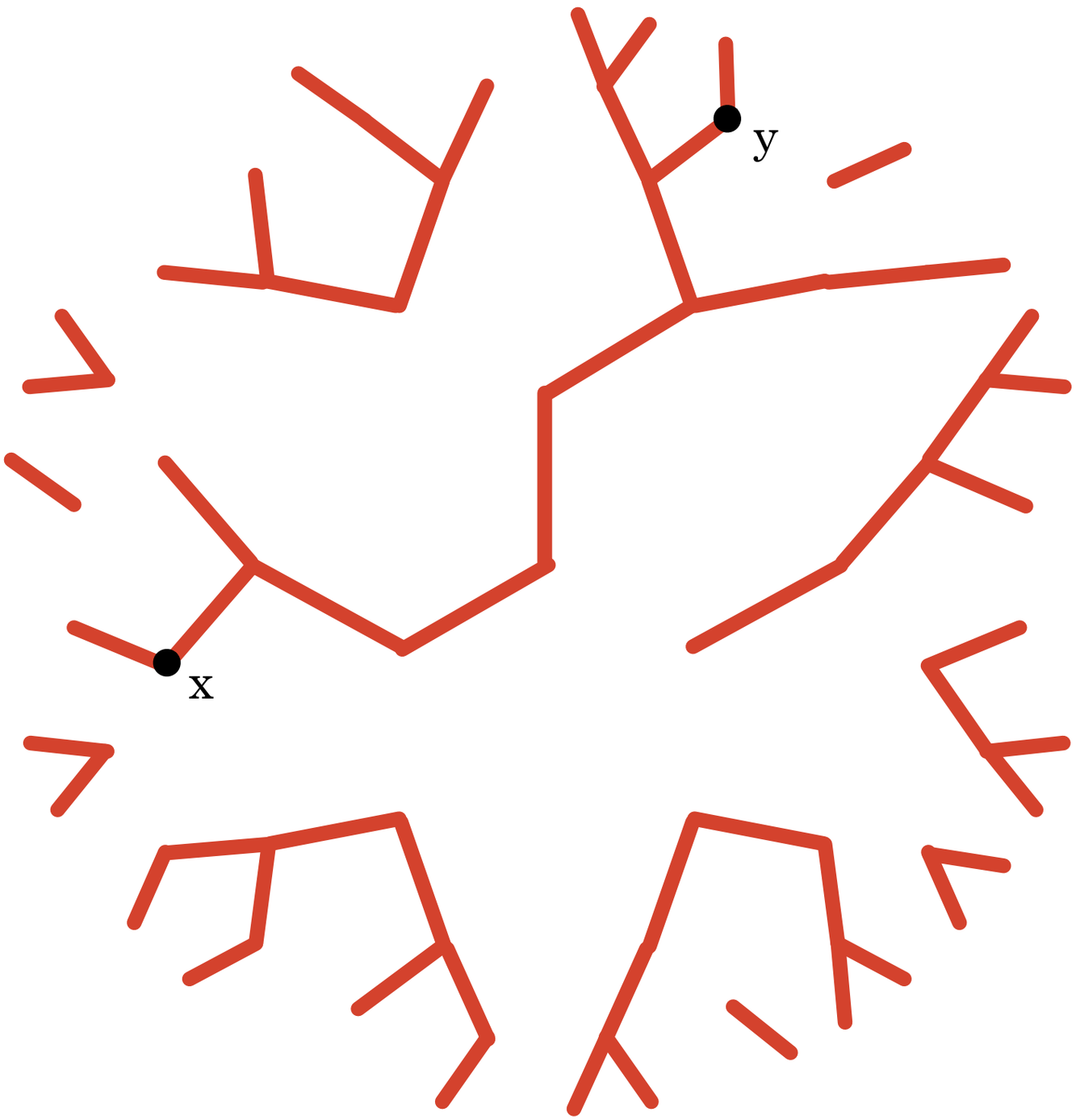
- Use wired boundary conditions
- MSF in bulk



Bethe Lattice MSTs

MST:

- Use wired boundary conditions
- MSF in bulk
- Q: Are two points connected in the bulk or via boundary?



Bethe Lattice MSTs

MST:

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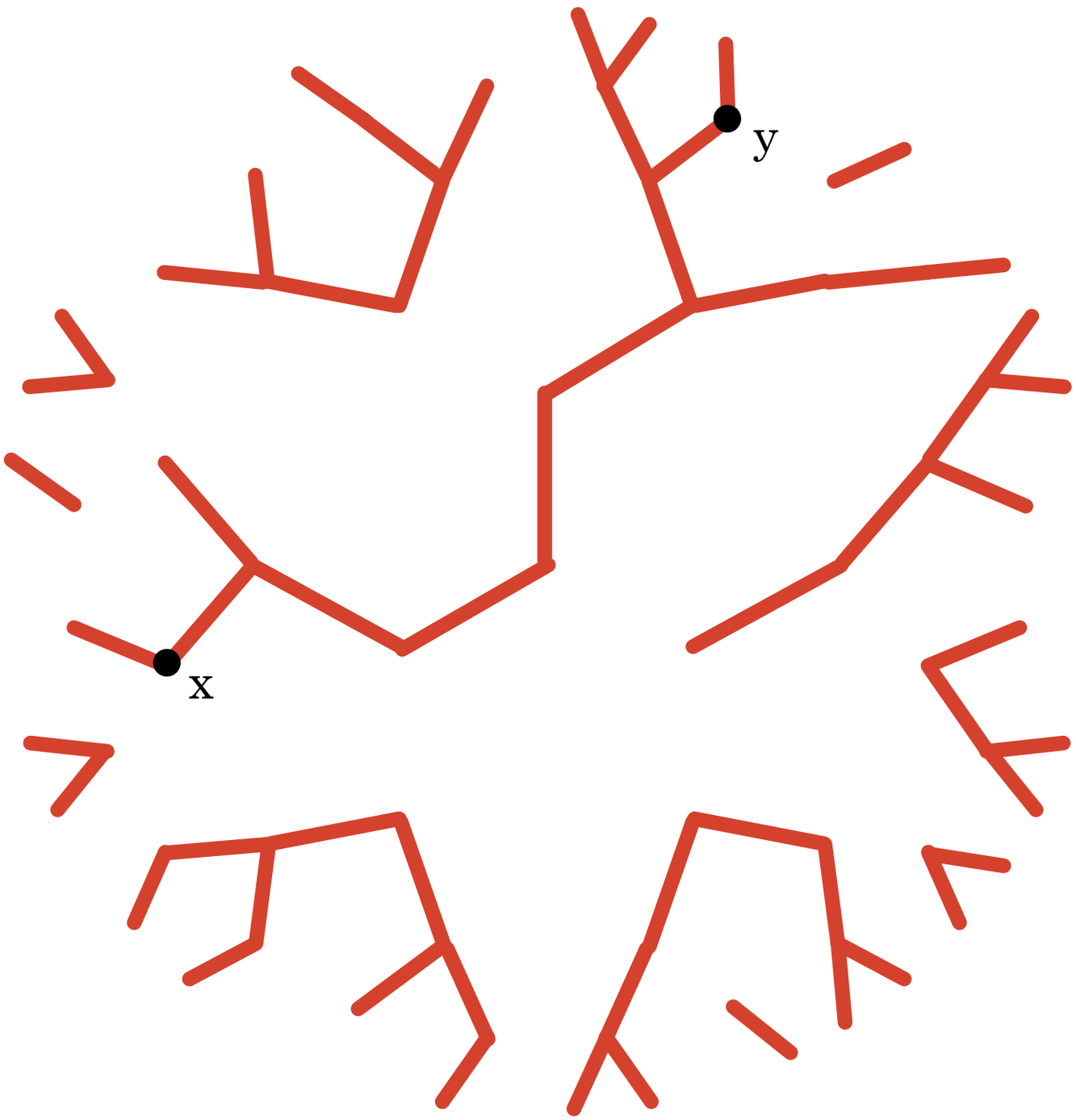

Bethe Lattice MSTs

MST:

- Use wired boundary conditions
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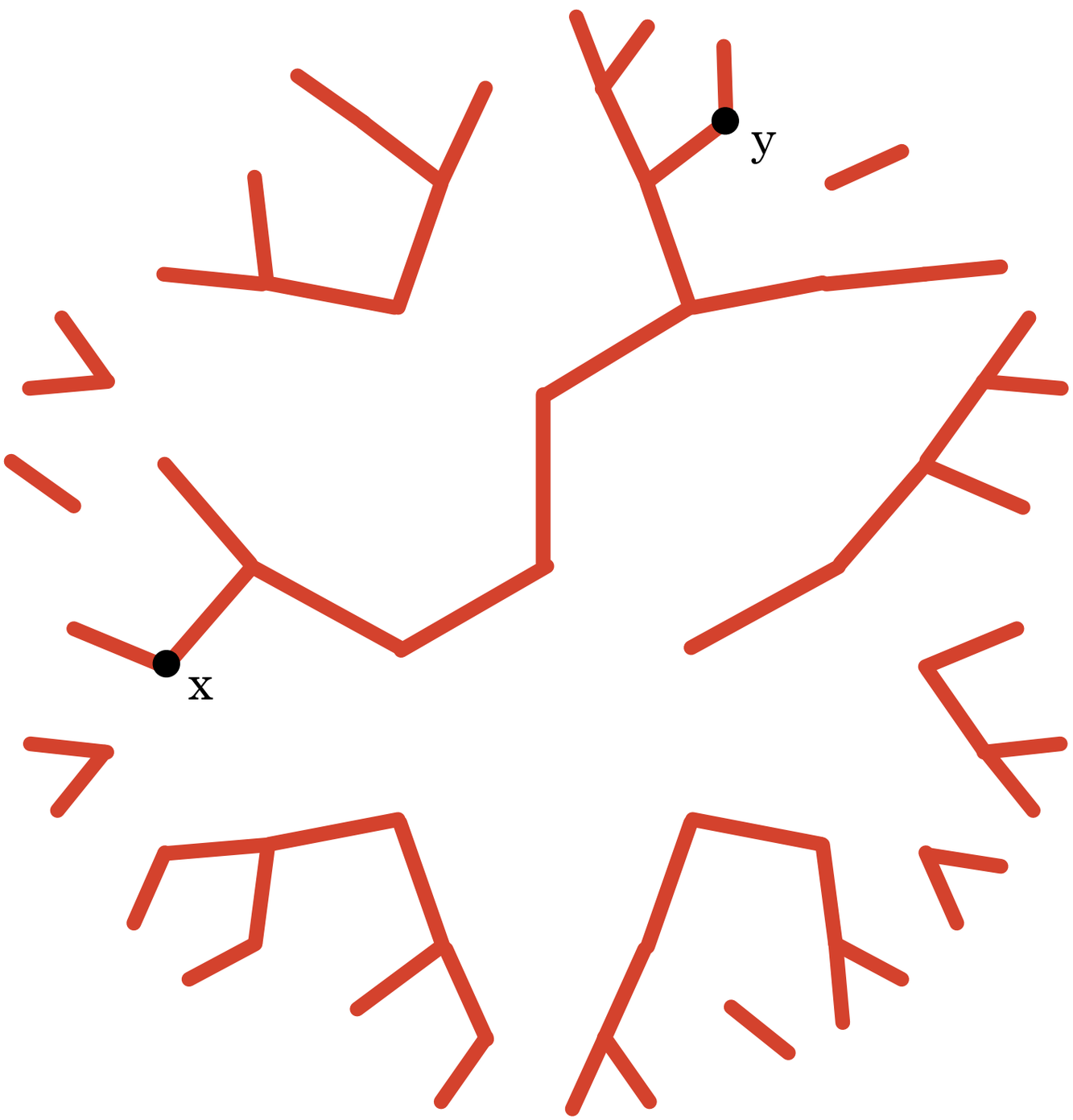
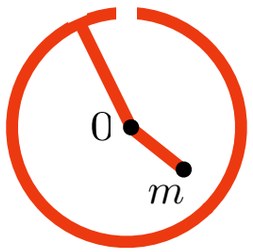
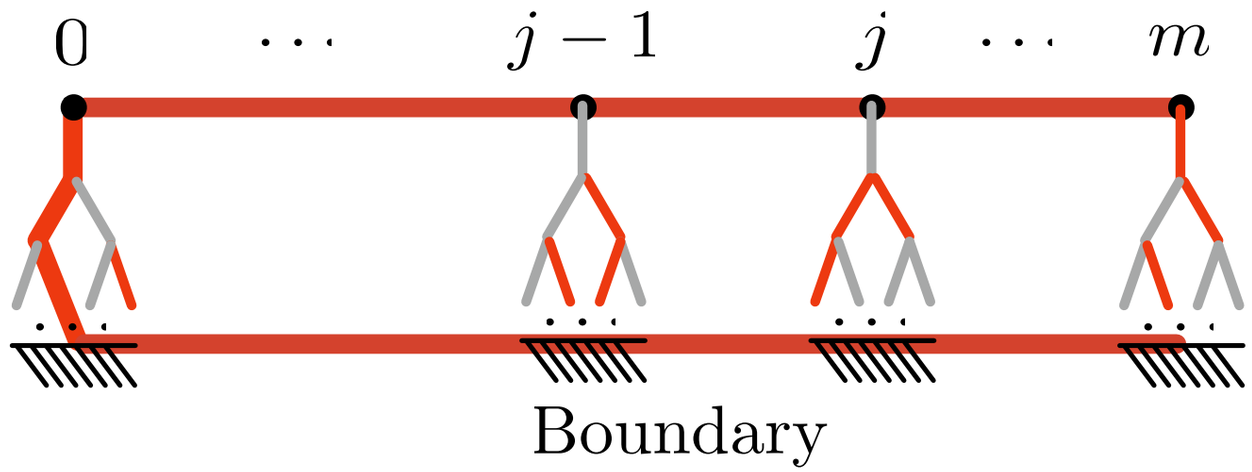
- Q: Are two points connected in the bulk or via boundary?
  - = Mean-field D of MSF clusters
  - = d_c where hyperscaling, MFT break down



MFT connectedness functions

Definitions:

$P_{(0;m)} = \Pr$ 

$= \Pr$ 

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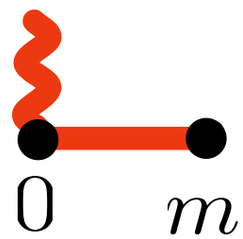
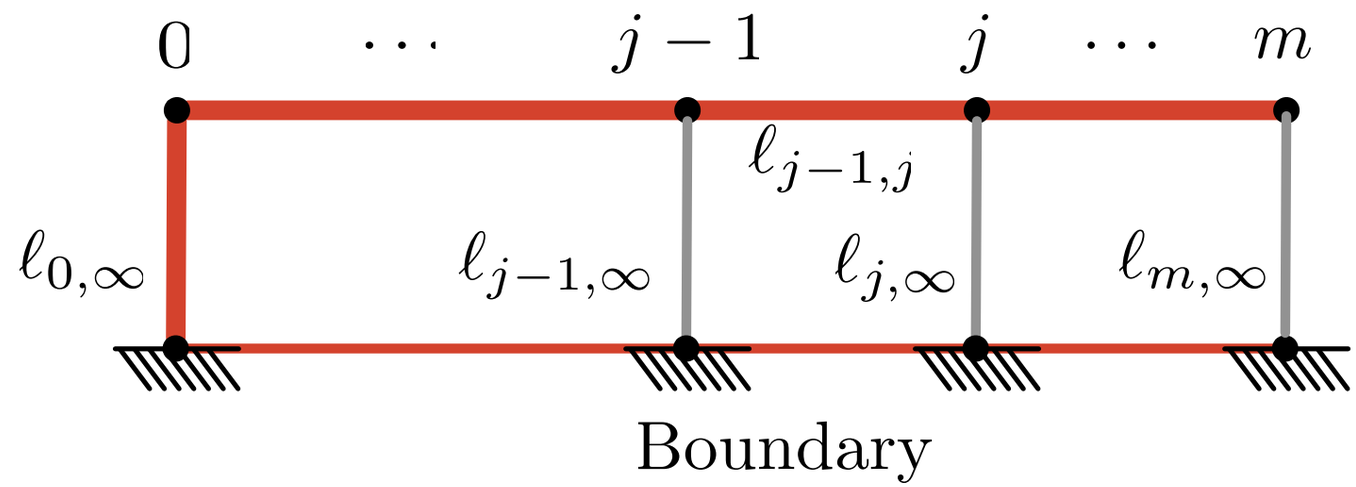
$P_{(0;m)} =$ 

$=$ 

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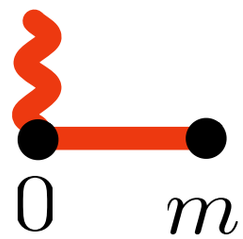
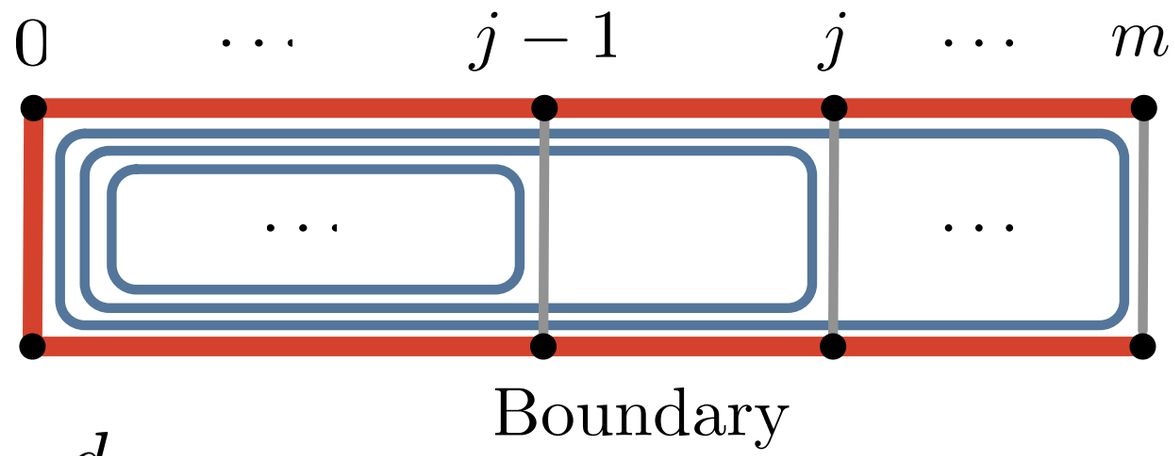
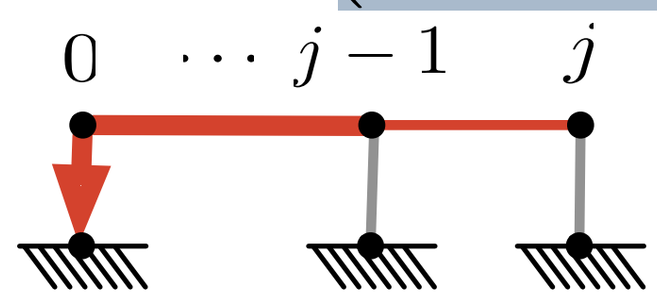
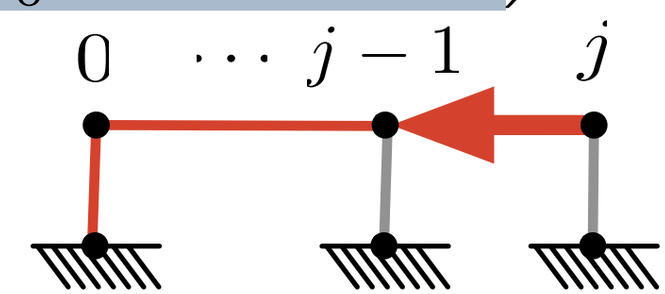
$P_{(0;m)} =$ 

$=$ 

Work with rates $\Phi_j(p) = \dfrac{d}{dp} P'_{(0;j)}(p)$

Recursion: $\Phi_j(p) \quad = \quad F(p)^{\sigma-1} \left(p\Phi_{j-1}(p) + \int_0^p dp' \, \Phi_{j-1}(p') \right)$

Either 

or 

$$= \int_0^1 dp' \, \underbrace{\left[F(p)^{\sigma-1} \frac{d}{dp} \left[p\theta(p-p') \right] \right]}_{\equiv K(p,p')} \Phi_{j-1}(p')$$

MFT connectedness functions

Connectedness functions via iterated kernel:

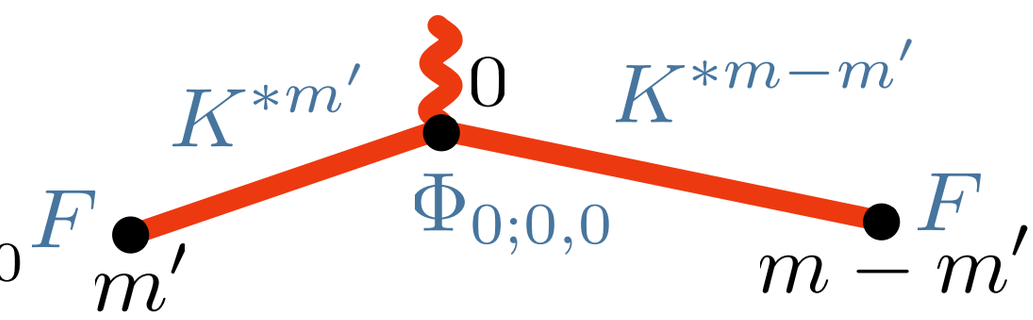
$$K^{*m}(p,p') = \int_0^1 dp_1 \cdots \int_0^1 dp_{m-1}\, K(p,p_1)\cdots K(p_{m-1},p')$$

= real-space propagator

Initial condition: $\Phi_0(p) = \dfrac{d}{dp}\left(1 - F(p)^\sigma\right)$ = $\Phi_0(p)$ 

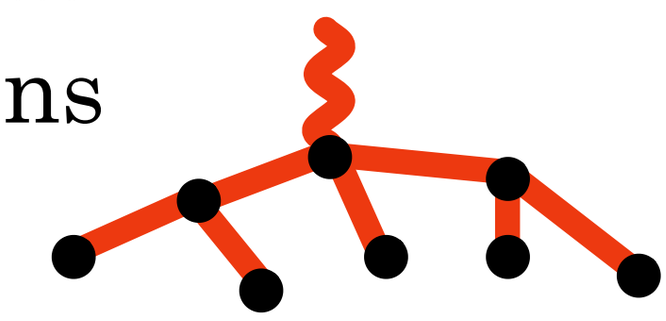
Then $P_{(0;m)}(p_0) = \displaystyle\int_0^{p_0} dp\, dp'\, \Phi_0(p')K^{*m}(p,p')F(p)$

and $C^{(2)}(m;p_0) = \displaystyle\sum_{m'=0}^{m}$



& similarly for n-point functions

= tree-level diagrams



Kernel asymptotics

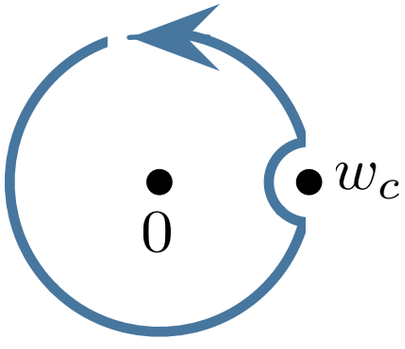
Generating function
"discrete Fourier-Laplace transform"

$$\widetilde{\Phi}_w(p) = \sum_{j=0}^{\infty} w^j \Phi_j(p)$$

Invert via Green's function $g_w(p,p')$
= momentum space propagator

$$\int_0^1 dp_2 \left[\delta(p_1 - p_2) - wK(p_1, p_2)\right] g_w(p_2, p_3) = \delta(p_1 - p_3)$$

$$K^{*m}(p_1, p_2) = \frac{1}{2\pi i} \oint dw \, \frac{g_w(p_1, p_2)}{w^{m+1}}$$



Extract large-m behavior from first pole encountered at w_c

$$1 - w_c p_2 F^{\sigma-1}(p_2) = 0$$

MFT scaling from lattice

"1-pt.": $P_{(0,m)} = \dfrac{P_\infty(p_0)}{(\sigma+1)\sigma^{m-1}} \to \sum_{\text{sites}}^{m} P_{(0,m')}(p_0) = mP_\infty(p_0)$

Compare generating function w/Fourier: $m \sim r^2$

$\Rightarrow$ Paths on MSF look like random walks

Asymptotic regime: $m \gg \mathcal{O}\big((p_0 - p_c)^{-1}\big)$ "correlation length"

2-pt.: $C^{(2)}(m;p_0) \sim \dfrac{1}{3}(\sigma\delta p_0)^2 m^2 \sigma^{-m} \to \sum_{\text{sites}}^{m} C^{(2)}(m';p_0) \sim m^3 \delta p_0^2$

$\Rightarrow$ MSF clusters have fractal dimension $D = 6$

$\Rightarrow$ $d_c = 6$ for MSTs

(k+1)-pt.: $\sum_{\text{sites}}^{m} C^{(k+1)} = \overline{M(m)^k} \sim m^{3k}$ $\Rightarrow$ not multifractal

Part II: Kruskal clusters at $d<d_c$, $p\leq p_c$

Only finite cycles

Diagrammatics

- Percolation = q → 1 Potts model; want *geometric* interpretation
- Return to low-*p* expansion for percolation on any finite graph (Essam '72, '80)
- Percolation: two points connected if there exists at least one path between them
  - Expand using inclusion-exclusion on paths
  - Group all subsets of paths w/same support → sum over *subgraphs*
  - Equivalent form using subsets of edges
  - Graphs carry associated diagrammatic weight identical to q → 1 Potts tensor contractions

Diagrammatics

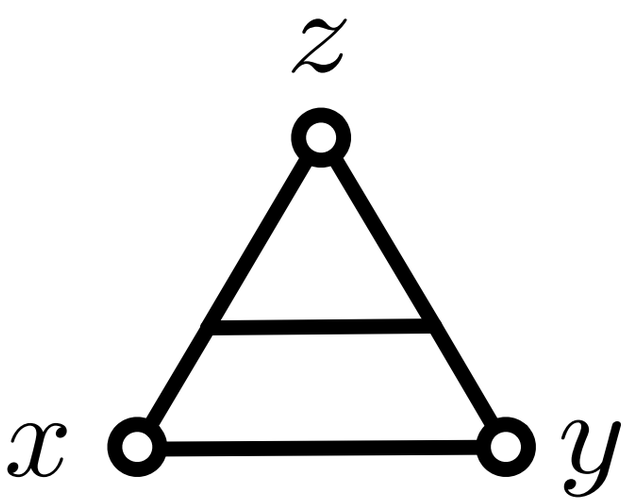
- MSF problem: want Pr[x,y connected by path through z]
- Can identify MST path between two points as minimum of ordering relation
  - Consequence of geodesic property → ultrametricity (transport applications)
  - Weight now depends on *barrier cost ordering* π
  - Inclusion-exclusion still applicable; must sum over orderings of each graph

$$\widetilde{C}^{\mathbf{z}}_{\mathbf{x},\mathbf{y}}(\ell_0) = \sum_{G \in G_{\mathbf{x},\mathbf{y},\mathbf{z}}} \Pr[G \leq \ell_0] \sum_{\pi \in S_{|E(G)|}} d_{\mathrm{MSF}}(G|\pi) \Pr[\pi],$$

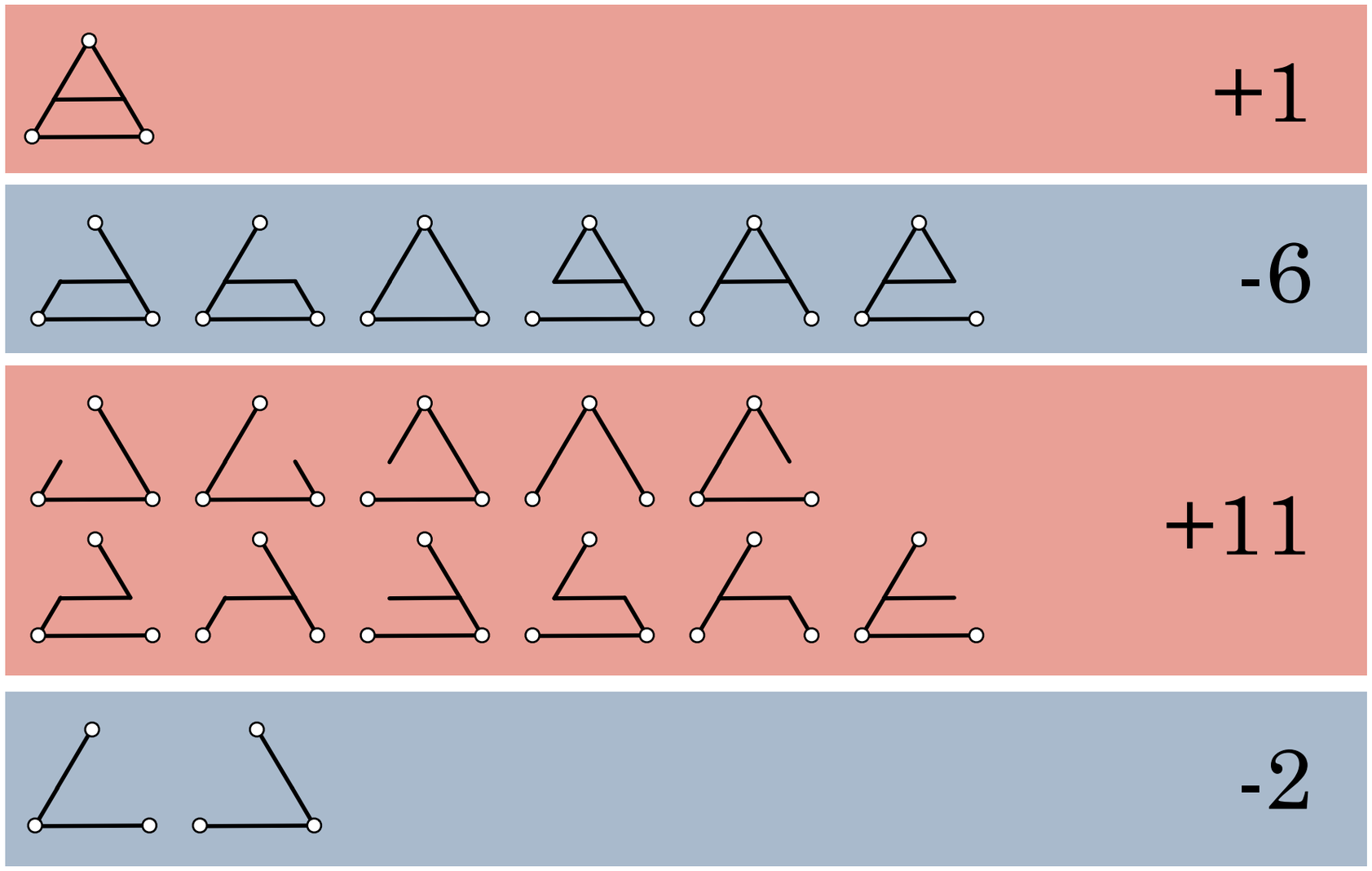
Example calculation

Percolation (3-pt)

$$d(G) = \sum_{E' \subseteq E} (-1)^{|E|-|E'|} \times \mathbb{I}\left[E' \text{ connects } x, y, z\right]$$



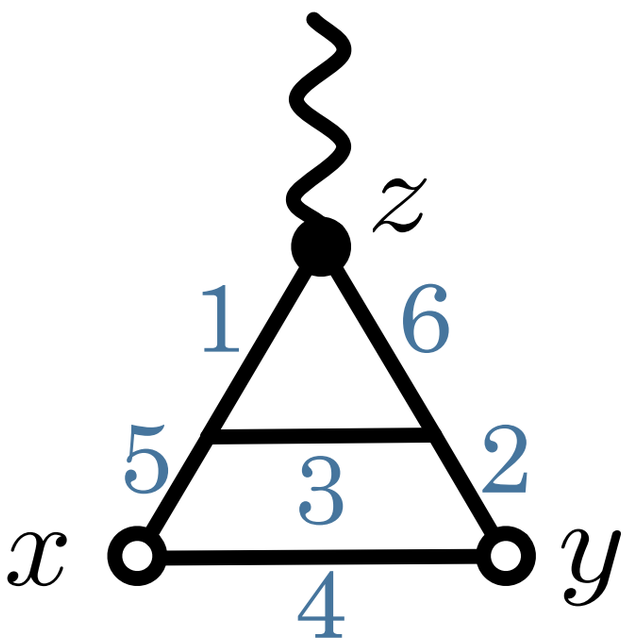
d(G) = 4



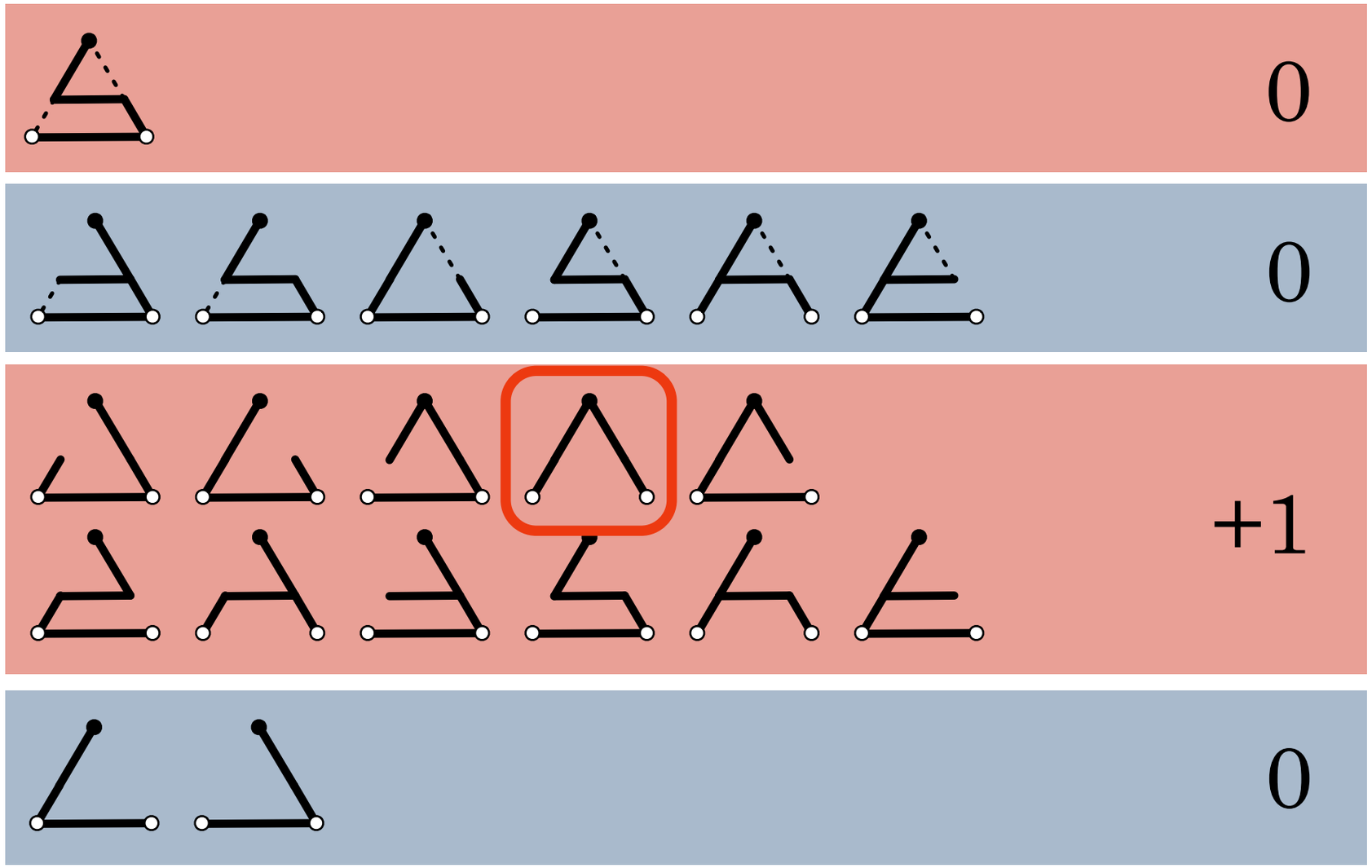
Example calculation

MSF path vertex

$$d_{\mathrm{MSF}}(G|\pi) = \sum_{E' \subseteq E} (-1)^{|E|-|E'|} \mathbb{I}[E' \text{ connects } x, y] \times \mathbb{I}[\gamma_{\mathrm{MST}}(G'_{E'}|\pi'_{E'}) \text{ passes through } z]$$



$d_{\mathrm{MSF}}(G|\pi) = 1$



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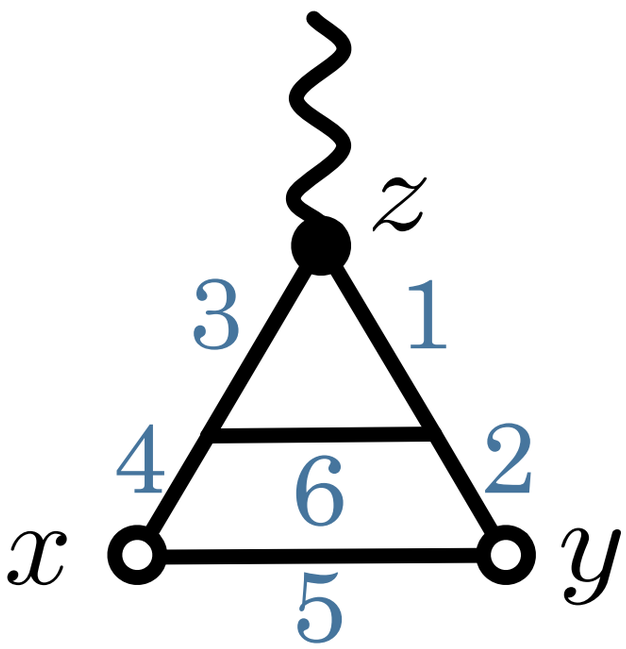
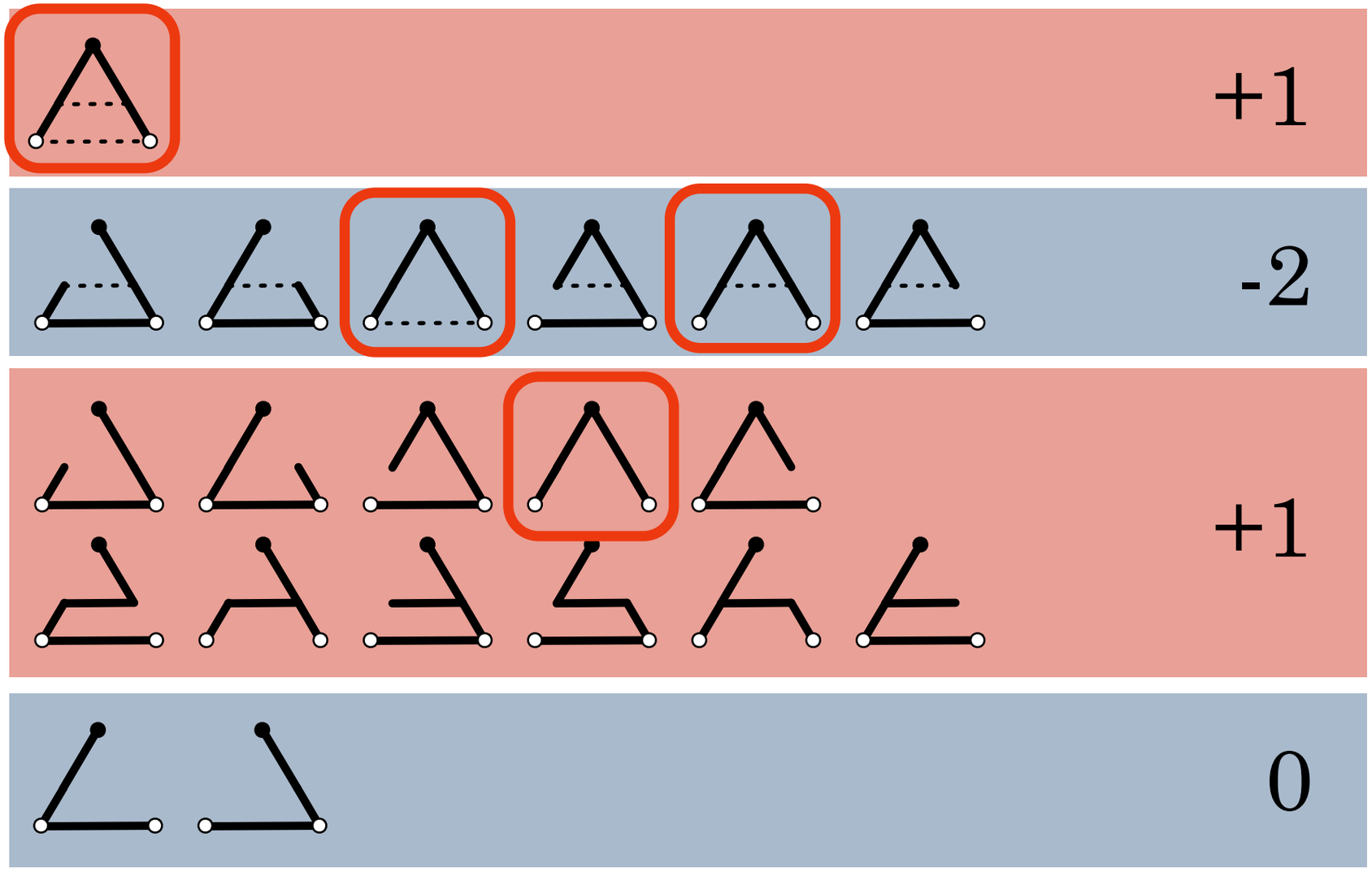


$d_{\mathrm{MSF}}(G|\pi') = 0$



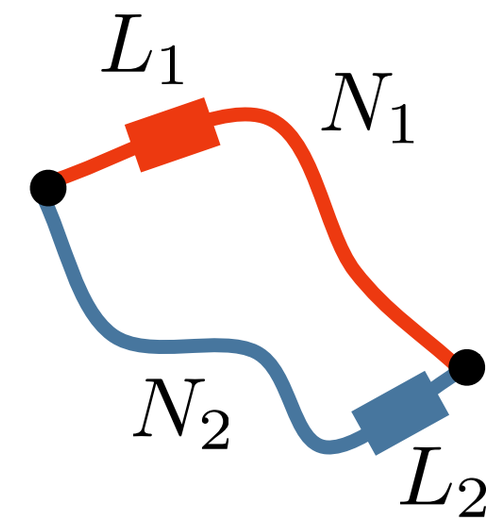
Lattice to continuum

Sum over graphs of same topology → continuum expansion

d(G), $d_{MSF}(G\,|\,\pi)$ invariant under insertion of paths of edges

Proper weight for orderings?

Toy Example:



Only need keep max. cost edge (*barrier*) along any path

$$\Pr[G \leq p_0] = \prod_{e \in E(G)} p_0 = p_0^{N_1} p_0^{N_2}$$

$$\Pr[L_1 < L_2 \leq p_0] = \int_0^{p_0} dp_2 \int_0^{p_2} dp_1 \frac{d}{dp_1}\frac{d}{dp_2} P_G(p_1, p_2)$$

$$= \frac{N_2}{N_1 + N_2} p_0^{N_1+N_2}$$

Generalization straightforward...

Continuum theory

- Generating functions of random walk = massive propagator for scalar field
  - Can neglect excluded volume constraint for RG
- Interpret mass on each propagator as MST barrier for given path
  - *Higher* barrier $p \to$ formed *later* as Kruskal is run $\to$ *smaller* value of t

$$\mathcal{O}_{\mathrm{MSF}}(\pi, t_0) = \int \cdots \int_{\infty > t_{\pi(1)} \geq t_{\pi(2)} \geq \cdots \geq t_{\pi(n)} \geq t_0} \prod_{\epsilon \in E} dt_\epsilon \frac{d}{dt_\epsilon}$$

acting on Feynman diagram of φ^3 theory

Continuum theory

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  - Can neglect excluded volume constraint for RG
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acting on Feynman diagram of φ^3 theory

- Diagrammatic weights unchanged

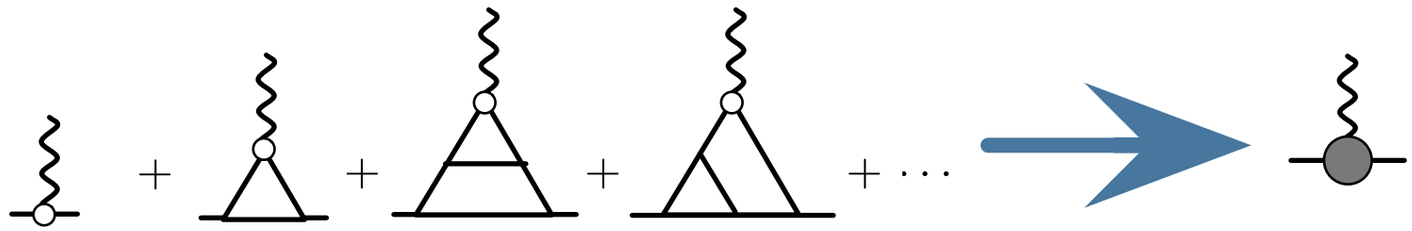
To summarize

- To generate the Feynman graph expansion for the MSF path vertex function,
  - Start with graphs for the mass insertion in a massive cubic theory;
  - For each graph and each ordering of edge masses,
    - compute $d_{\mathrm{MSF}}(G\,|\,\pi)$ for that ordering,
    - assign each edge a unique mass and act with $O_{\mathrm{MSF}}(t_0)$ for that ordering and
    - sum over orderings and perform integration over loop momenta of Feynman diagram.

$$\widetilde{C}^{z}_{x,y}(p_0) \to \sum_{\mathcal{G}} \sum_{\pi' \in S_{|E|}} d_{\mathrm{MSF}}(\mathcal{G}|\pi')\mathcal{O}_{\mathrm{MSF}}(\pi', t_0) I(\mathcal{G}, \{t_\epsilon\})$$

“Renormalizability”

- The (possible) problem:

$\Gamma^{PV} =$ 

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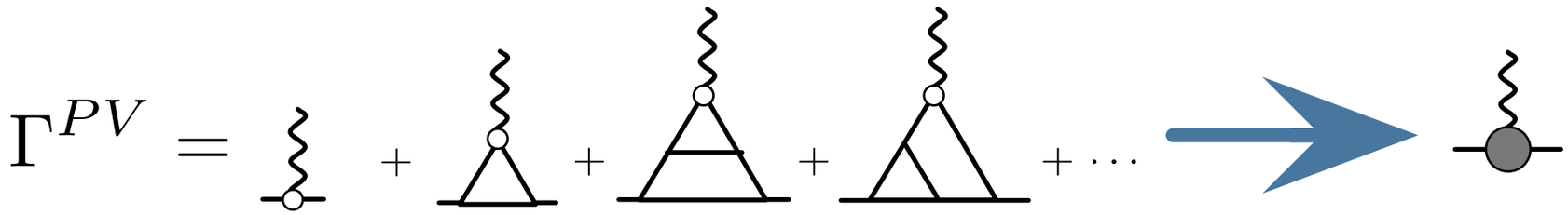

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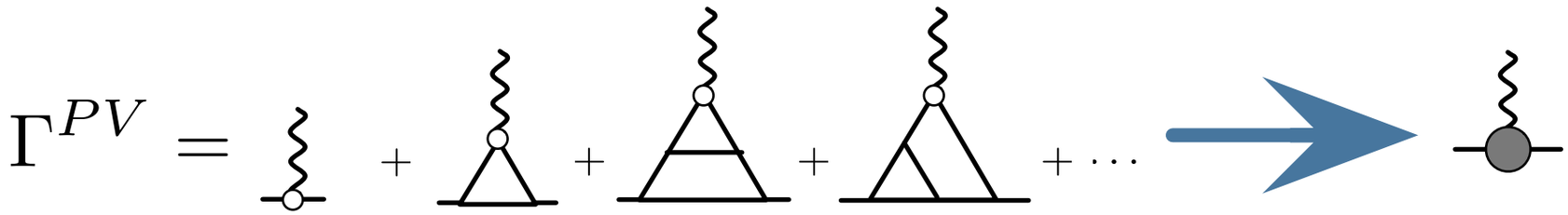

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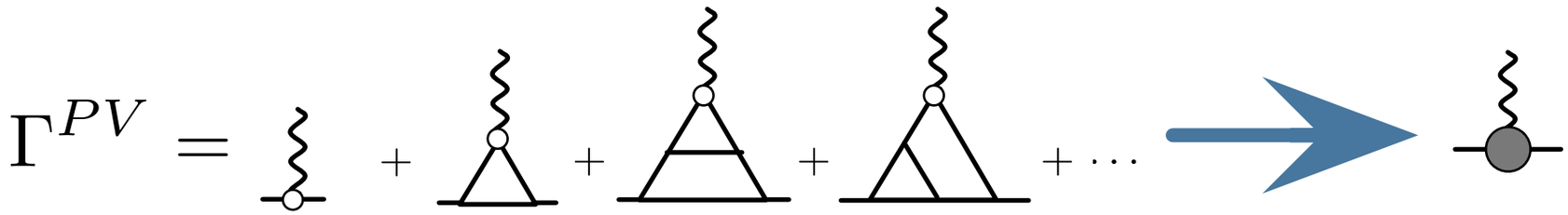

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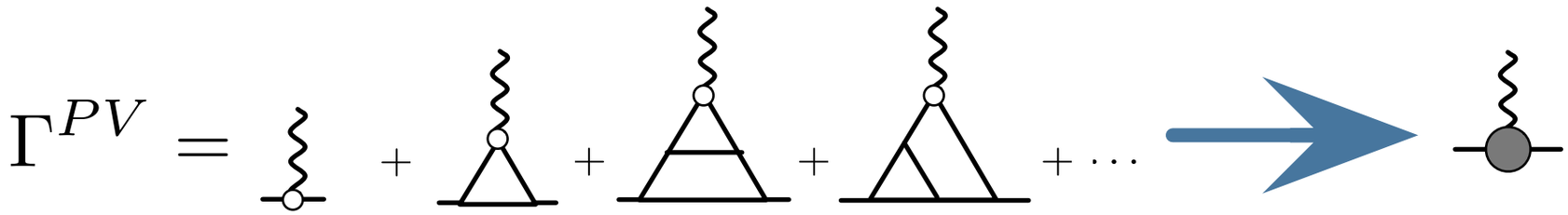

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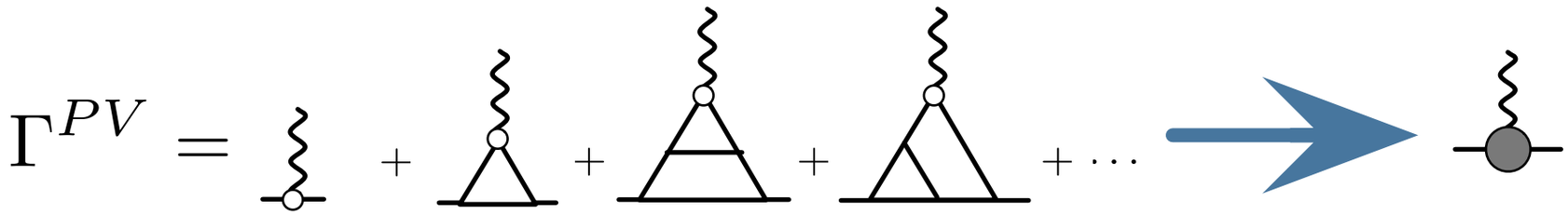

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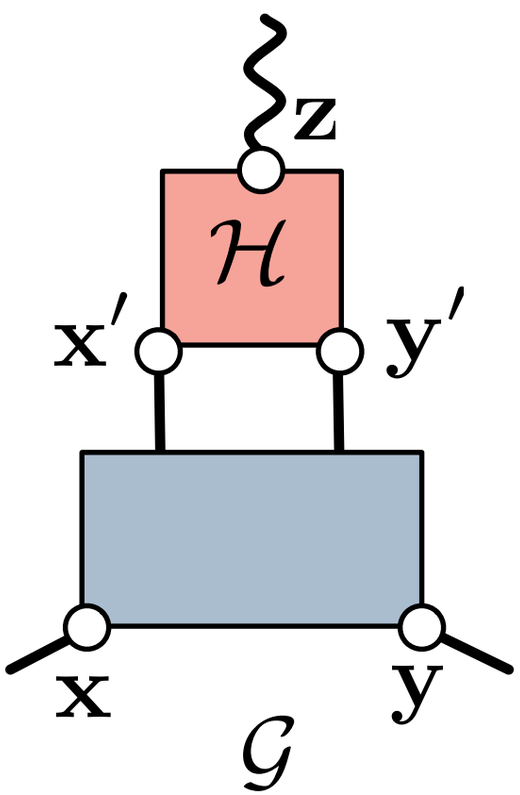

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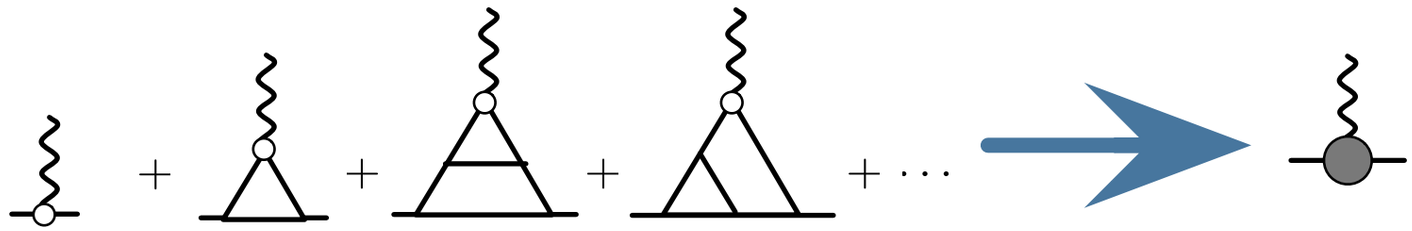
$\Gamma^{PV} =$ 

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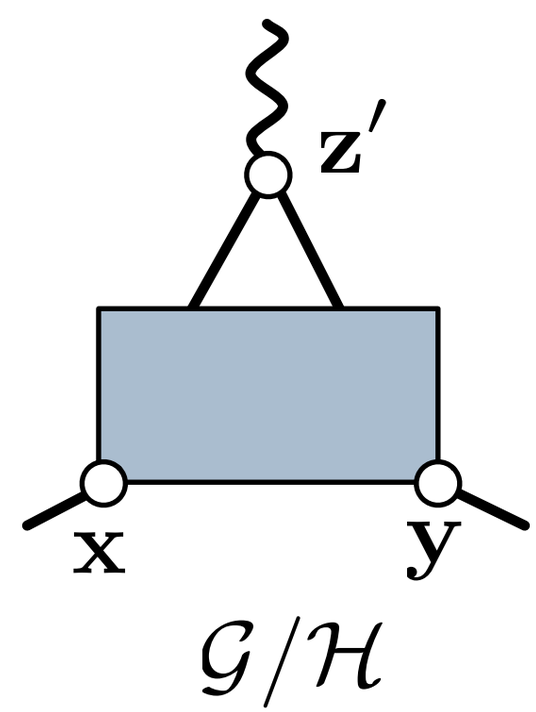
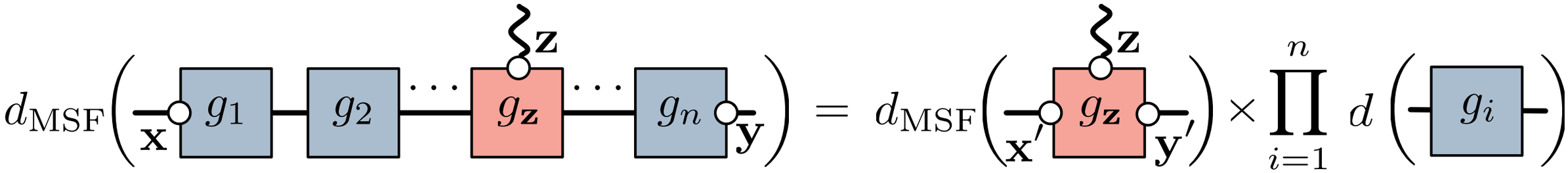


“Renormalizability”

- O_{MSF} factorizes
- Only UV-divergent contributions to any diagram come from a *subset* of all orderings
  - Short paths form first → High-momentum propagators get higher masses
- d_{MSF} factorizes for these orderings (*not* in general)
  - Two-point diagram gives barrier mass insertion

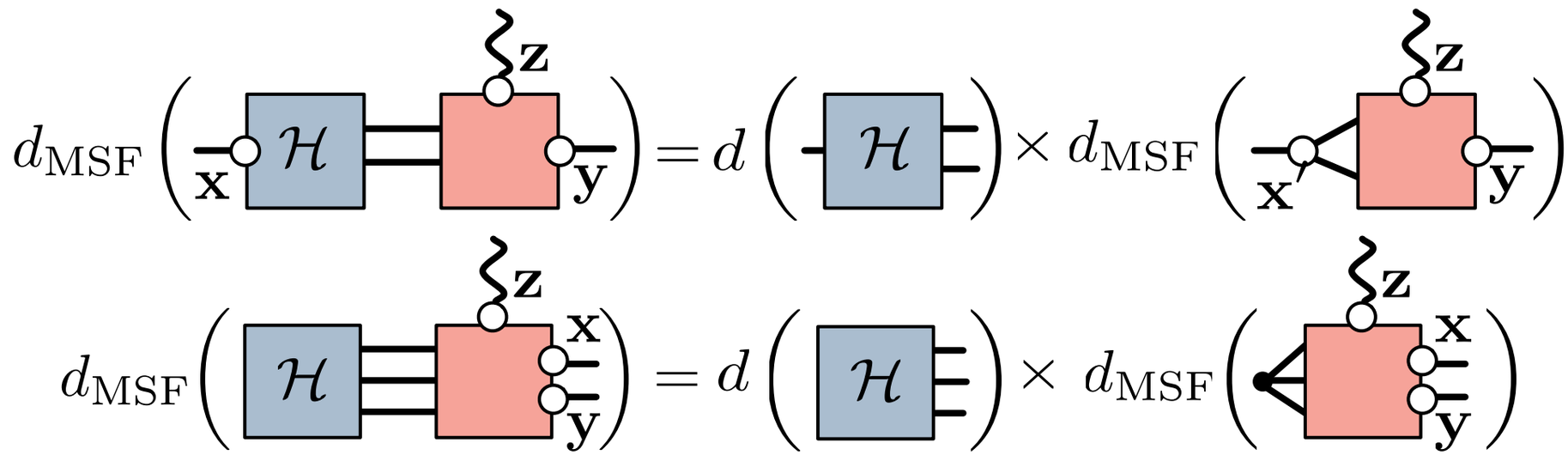
Factorization of d_{MSF}

(RG-relevant)

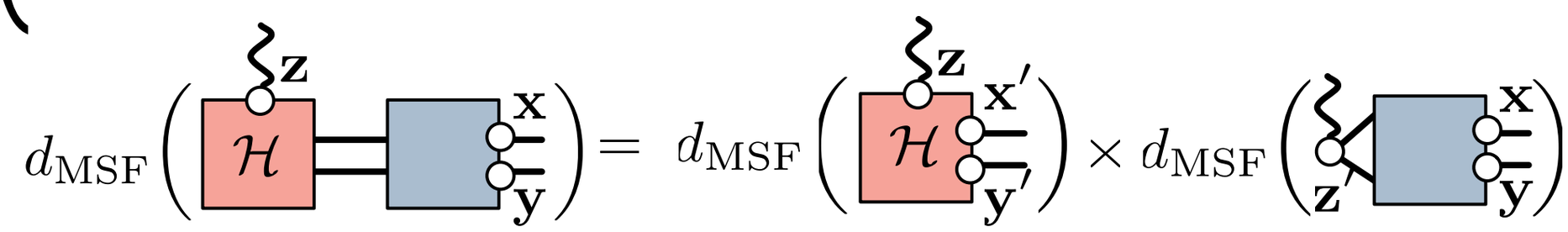


(1PI)

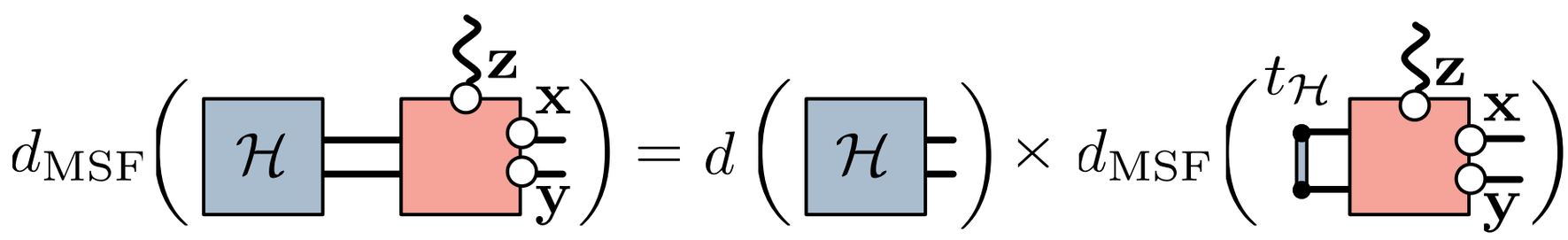
Percolation



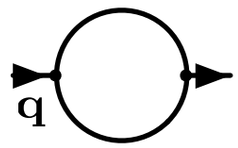
Path vertex:



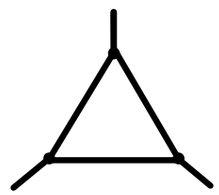
Barrier:



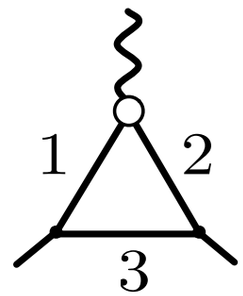
1 loop RG



$d_2 = -1$



$d_3 = -2$



$$d_{\mathrm{MSF}}(\triangle | t_1, t_2 > t_3) = 0$$
$$\text{else } = -1$$
$$\Rightarrow d_{\mathrm{PV}} = -\frac{2}{3}$$

$$\beta(u_0) = -\frac{\varepsilon}{2} u_0 + \left(\frac{d_2}{4} - d_3\right) u_0^3 + \mathcal{O}(u_0^5, \varepsilon u_0^3)$$

$$\gamma_\phi(u_0) = \frac{d_2}{6} u_0^2 + \mathcal{O}(u_0^4, \varepsilon u_0^2) \qquad (\varepsilon = d_c - d)$$

$$\gamma_{PV}(u_0) = \left(d_{PV} - \frac{d_2}{6}\right) u_0^2 + \mathcal{O}(u_0^4, \varepsilon u_0^2)$$

$$D_{\mathrm{MSF}} = 2 - \frac{(6 d_{PV} - d_2)}{3(d_2 - 4 d_3)} \varepsilon = 2 - \frac{\varepsilon}{7} + \mathcal{O}(\varepsilon^2)$$

singly connected bonds: $D_{\mathrm{perc}} = \nu_{\mathrm{perc}}^{-1} = 2 - \frac{5\varepsilon}{21} + \mathcal{O}(\varepsilon^2)$

Future directions

- Behavior for $p > p_c$
- Application to other problems?
  - Loop-erased random walk from self-avoiding walk
  - Two-replica formulation of $\pm J$ EA spin glass
    - Metastable states???
    - Ground state enumeration?
- Perturbation away from strong disorder
  - Failure of greedy algorithms related to, e.g., RSB instability?

Summary

- MST problem has $d_c = 6$, not 8
- $D_{\mathrm{MST}} = 2 - \frac{\varepsilon}{7} + \mathcal{O}(\varepsilon^2)$
- Treatment of quenched disorder in strong limit without (explicit?) replicas, cavity method, FRG, supersymmetry,...
  - Nonlocality → perturbation series not generated by (local) action
- Simplification within RG framework

Part I: TSJ and N. Read, arXiv:0902.3651
Part II: TSJ and N. Read, arXiv:0909.5343