Lifshitz theories in higher spin gravity

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Higher spin theories in $d \geq 3$ by Vasiliev et al:

\begin{align*}
    dW &= W \wedge \ast W, \\
    dB &= W \ast B - B \ast W \\
    dS &= W \ast S - S \ast W \\
    S \ast S &= C + R(B)
\end{align*}

- AdS/CFT duality to Large N vector model (such as $O(N)$ in $d=3$)
- Infinite tower of higher spin fields
- Nonlinear coupling of matter
- Nonlocal (infinite number of derivatives)
- No action, only equations of motion
- Mainly classical, hard to quantize
Review of 3-dim higher spin gravity

Situation in $d = 3$ somewhat simpler:

- Infinite tower of higher spins coupled to scalar matter
- Linearizing scalar: theory can be reformulated as $hs(\lambda) \times hs(\lambda)$ Chern-Simons gauge theory
- Gaberdiel and Gopakumar: Theory dual in a ’t Hooft like limit to $W_N$ minimal model coset
- For $\lambda = \pm N$ theory truncates to $SL(N, R) \times SL(N, R)$ Chern-Simons gauge theory
- Interacting theory of spin 2,3,...,N
Review of 3-dim higher spin gravity

Chern-Simons action at level $k$ and $-k$ and gauge group $SL(N, R) \times SL(N, R)$

$$S = S_{CS}[A] - S_{CS}[ar{A}]$$

where

$$S_{CS}[A] = \frac{k}{4\pi} \int \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

For $SL(3, R) \times SL(3, R)$, theory contains spin 2 and spin 3 field

$$e_\mu = \frac{1}{2} (A_\mu - \bar{A}_\mu), \quad \omega_\mu = \frac{1}{2} (A_\mu + \bar{A}_\mu)$$

with

$$g_{\mu\nu} = \frac{1}{2} \text{tr}(e_\mu e_\nu), \quad \phi_{\mu\nu\rho} = \frac{1}{6} \text{tr}(e_{(\mu} e_{\nu} e_{\rho)}).$$
Holographic Lifshitz space-times

- Lifshitz symmetry: anisotropic scaling wrt. space and time

\[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x} \]

- in two dimensions only one spatial coord \( x \) and time translations \( H \), spatial translations \( P \) and Lifshitz scaling \( D \) form Lifshitz algebra

\[
[P, H] = 0 \quad [D, H] = zH \quad [D, P] = P
\]

- the energy density \( \mathcal{E} \), the energy flux \( \mathcal{E}^x \), the momentum density \( \mathcal{P}_x \) and the stress energy tensor \( \Pi_x^x \)

\[
\partial_t \mathcal{E} + \partial_x \mathcal{E}^x = 0, \quad \partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0, \quad z\mathcal{E} + \Pi_x^x = 0
\]
Holographic Lifshitz space-times

- Holographic realization of Lifshitz

\[ ds^2 = L^2 \left( d\rho^2 - e^{2z\rho} dt^2 + e^{2\rho} dx^2 \right) \]

Shift \( \rho \rightarrow \rho + \ln \lambda \) induces Lifshitz scaling on \( t, x \) coordinates.

- For \( SL(3, R) \times SL(3, R) \) CS gravity. Generators of \( SL(3, R) \)

\[ SL(2, R) : \quad L_{\pm 1}, L_0, \quad \text{spin} 2 : \quad W_{\pm 2}, W_{\pm 1}, W_0 \]

- Radial gauge fixes \( \rho \) dependence with \( b = \exp(\rho L_0) \)

\[ A_\mu = b^{-1} a_\mu b + b^{-1} \partial_\mu b, \quad \tilde{A}_\mu = b \tilde{a}_\mu b^{-1} + b \partial_\mu (b^{-1}) \]
Holographic Lifshitz space-times

- The following connection gives $z = 2$ Lifshitz metric.

\[ a = W_2 \, dt + L_1 \, dx, \quad \bar{a} = W_{-2} \, dt + L_{-1} \, dx \]

Define an asymptotic Lifshitz connection

\[ A - A_{\text{Lif}} \sim \mathcal{O}(1), \quad \text{as } \rho \to \infty \]

Two goals:

- Identify the Lifshitz stress-energy complex in asymptotic Lifshitz connection

- Realize Lifshitz symmetries as residual gauge transformations on asymptotically Lifshitz connections.
Holographic Lifshitz theories

Holographic Lifshitz space-times

- Asymptotic Lifshitz + flatness ($F = 0$) connection of the form
  
  \[ a_t = W_2 - 2\mathcal{L}W_0 + \frac{2}{3}\mathcal{L}'W_{-1} - 2\mathcal{W}L_{-1} + \left(\mathcal{L}^2 - \frac{1}{6}\mathcal{L}''\right)W_{-2}, \]
  \[ a_x = L_1 - \mathcal{L}L_{-1} + \mathcal{W}\mathcal{W}_{-2} \]

- Time evolution equation
  
  \[ \dot{\mathcal{L}} = 2\mathcal{W}', \quad \dot{\mathcal{W}} = \frac{4}{3}(\mathcal{L}^2)' - \frac{1}{6}\mathcal{L}''' \]
Holographic Lifshitz space-times

- em-complex for Lifshitz can be identified with

\[
\mathcal{E} = \mathcal{W} + \bar{\mathcal{W}}, \\
\mathcal{P}_x = \mathcal{L} - \bar{\mathcal{L}}, \\
\Pi^x_x = -2\mathcal{W} - 2\bar{\mathcal{W}}, \\
\mathcal{E}^x = -\left(\frac{4}{3} \mathcal{L}^2 - \frac{1}{6} \partial_x^2 \mathcal{L}\right) + \left(\frac{4}{3} \bar{\mathcal{L}}^2 - \frac{1}{6} \partial_x^2 \bar{\mathcal{L}}\right)
\]

- Where the energy density \(\mathcal{E}\), the energy flux \(\mathcal{E}^x\), the momentum density \(\mathcal{P}_x\) and the stress energy tensor \(\Pi^x_x\), with \(z = 2\).

\[
\partial_t \mathcal{E} + \partial_x \mathcal{E}^x = 0, \quad \partial_t \mathcal{P}_x + \partial_x \Pi^x_x = 0, \quad z\mathcal{E} + \Pi^x_x = 0
\]
Holographic Lifshitz space-times

- Time evolution equation for $\mathcal{L}, \mathcal{W}$ related to Boussinesq equation (eliminate $\mathcal{W}$)

$$
\ddot{\mathcal{L}} = \frac{8}{3} (\mathcal{L}^2)'' - \frac{1}{3} \mathcal{L}'''
$$

- Integrable system (bi-Hamiltonian) related to $W_3$ algebra (as KdV is related to Virasoro)

- Infinitely many conserved commuting charges

$$
q_1 = \int dx \ \mathcal{W}, \quad q_2 = \int dx \ \mathcal{L}
$$

$$
q_3 = \int dx \ \mathcal{W} \mathcal{L}, \quad q_4 = \int dx \ \left( \mathcal{W}^2 + \frac{4}{9} \mathcal{L}^2 + \frac{1}{12} (\mathcal{L}')^2 \right), \ldots
$$
Realization of Lifshitz symmetries

- Symmetries realized as gauge transformations leaving asymptotic Lifshitz form invariant
- Variation of charge

\[ \delta Q(\Lambda) = -\frac{k}{2\pi} \int_{-\infty}^{\infty} dx \text{tr}(\Lambda \delta A_x) \]

if integrable leads to charge \( Q(\Lambda) \)
- Algebra of charges

\[ \{ Q(\Lambda), Q(\Gamma) \} = \delta_\Lambda Q(\Gamma) \]
Realization of Lifshitz symmetries

- Charges

\[
Q(\Lambda_H) = \frac{2k}{\pi} \int_{-\infty}^{\infty} dx \mathcal{W}
\]
\[
Q(\Lambda_P) = \frac{2k}{\pi} \int_{-\infty}^{\infty} dx \mathcal{L}
\]
\[
Q(\Lambda_D) = -\frac{2k}{\pi} \int_{-\infty}^{\infty} dx (2t\mathcal{W} + x\mathcal{L})
\]

- Satisfy Lifshitz algebra

\[
\{ Q(\Lambda_H), Q(\Lambda_P) \} = 0,
\]
\[
\{ Q(\Lambda_D), Q(\Lambda_H) \} = 2Q(\Lambda_H),
\]
\[
\{ Q(\Lambda_D), Q(\Lambda_P) \} = Q(\Lambda_P).
\]

- Open question: infinite dimensional extension?
Higher spin black holes

- Black hole in higher spin gravity: Geometric characterization difficult as higher spin gauge transformations act on metric
- Horizon, causal structure are gauge dependent
- Higher spin BH for asymptotic AdS (in radial gauge)

\[ a_+ = (L_1 + \mathcal{L}L_{-1} + \mathcal{W}_W_{-2}), \quad a_- = \mu(W_2 + \cdots) \]

- Gauge invariant characterization of BH: holonomy around euclidean time circle is equal to BTZ, i.e. in the center of SL(3,R).
Lifshitz higher spin black holes

- For Lifshitz asymptotics replace l.c. coordinates $+ \rightarrow x, - \rightarrow t$.

$$a_x = (L_1 + \mathcal{L}L_{-1} + \mathcal{W}_2), \quad a_t = \mu_2 \mathcal{W}_2 + \mu_1 L_1 + \cdots$$

- Nonrotating BH: $\bar{a}$ determined by $a$

$$\bar{a}_x = -a_x^T, \quad \bar{a}_t = a_t^T$$

- Interpretation of $\mathcal{L}, \mathcal{W}$ from em-complex, $\mu_1, \mu_2$ chemical potential.

- Role changed from AdS BH (energy $\sim \mathcal{W}, \beta \sim \mu_2$)
Lifshitz Black holes

Lifshitz higher spin black holes

- Holonomy around euclidean time circle, taken the same as for BTZ

\[ \mathcal{P} \exp \left( \int dt A_t \right) = \text{diag}(e^{2\pi i}, 1, e^{-2\pi i}) \]

- Holonomy conditions: "equation of state" relate $\mathcal{W}, \mathcal{L}$ to $\mu_1, \mu_2$.

\[
0 = 3\mathcal{L}\mu_1^2 + 9\mathcal{W}\mu_1\mu_2 + 4\mathcal{L}^2\mu_2^2 - \frac{3}{4},
\]

\[
0 = 108\mathcal{W}^2\mu_2^3 + 8\mathcal{L}^2\mu_2 \left( 9\mu_1^2 - 4\mathcal{L}\mu_2^2 \right) + 27\mathcal{W} \left( \mu_1^3 + 4\mathcal{L}\mu_1\mu_2^2 \right).
\]
Thermodynamics

- On shell CS-action $I_0$ reduces to boundary term.
  \[ I_0^{\text{os}} = -4k(2L\mu_1 + 3W\mu_2) \]

- Add a boundary term to produce an euclidean action
  \[ I_1 = -8k(\mu_1 L + 2\mu_2 W) \]

- Has a good variational principle
  \[ \delta I_1 = 8k(L\delta\mu_1 + W\delta\mu_2) \]

- Using the holonomy conditions
  \[ \frac{\partial I_1}{\partial \mu_1} = 8kL, \quad \frac{\partial I_1}{\partial \mu_2} = 8kW. \]

- $W$, $L$ are charges and $\mu_2$, $\mu_1$ conjugate chemical potentials
grand potential is defined in terms of partition function

\[ \Phi = -\frac{1}{\beta} \ln Z = \frac{1}{\beta} l_1 \]

natural variables: temperature $T$ and chemical potential $\alpha$

\[ \mu_1 = \beta \alpha = \frac{1}{T} \alpha, \quad \mu_2 = \beta = \frac{1}{T} \]

$\beta$ multiplies $a_t$ and hence $g_{tt}$ and can be absorbed in periodicity of euclidean time.
Thermodynamics

- grand potential

\[ \Phi = -8k\left(\alpha\mathcal{L} + 2\mathcal{W}\right) \]

- Thermodynamic differential

\[ d\Phi = -SdT - Qd\alpha \]

- Determines charge and entropy

\[ Q = -\left.\frac{\partial \Phi}{\partial \alpha}\right|_T = -8k\mathcal{L} \]
\[ S = -\left.\frac{\partial \Phi}{\partial T}\right|_\alpha = \frac{1}{T}8k\left(2\alpha\mathcal{L} + 3\mathcal{W}\right) \]
Branches

- Charges $\mathcal{L}, \mathcal{W}$: extensive. Chemical potentials: $\mu_1, \mu_2$ intensive.
- Solution of the holonomy conditions:

$$0 = 3\mathcal{L}\mu_1^2 + 9\mathcal{W}\mu_1\mu_2 + 4\mathcal{L}^2\mu_2^2 - \frac{3}{4},$$

$$0 = 108\mathcal{W}^2\mu_2^3 + 8\mathcal{L}^2\mu_2 (9\mu_1^2 - 4\mathcal{L}\mu_2^2) + 27\mathcal{W} (\mu_1^3 + 4\mathcal{L}\mu_1\mu_2^2).$$

Has different branches: entropy, energy, grand potential all take different form.
Physical conditions decide whether a branch is sensible or not:

- Temperature $T$ positive
- Entropy $S$ positive
- $S$ has minimum for $T \rightarrow 0$
- Local thermodynamical stability
- In metric formulation a "black hole gauge exits" which displays a regular horizon

Branches: Solve $\mathcal{L}, \mathcal{W}$ in terms of $T, \alpha$ has four branches or vice versa has six branches.
Thermodynamics as function of $T$ for fixed $\alpha$

(a) Entropies.

(b) Grand potentials.

(c) $\mathcal{L}$ charge.

(d) $\mathcal{W}$ charge.
Branches

- Branch I satisfies all criteria to be sensible
- Other branches fail in one or more
- Same conclusion if we analyze branches in terms of extensive variables $\mathcal{L}, W$.  

Summary

- Asymptotic Lifshitz space times: Construction of stress-energy complex and Lifshitz symmetries in the CS formulation
- Construction of BH: very similar to AdS higher spin black holes \( \mu = \pm \) get replaced by \( t, x \)
- Identification of charges (energy, higher spin charge) and chemical potentials (temperature, \( \alpha \)) different
- Holonomy conditions have multiple branches, one physically sensible.
Generalizations

- CFT realization (?): $H = W_0 + \bar{W}_0$ instead of $L_0 + \bar{L}_0$.
- Infinite dimensional asymptotic symmetry generalizing Lifshitz scaling. Infinite set of commuting charges, conserved charges

$$q_3 = \int dx \nabla L, \quad q_4 = \int dx \left( \nabla^2 + \frac{4}{9} L^2 + \frac{1}{12} (L')^2 \right), \ldots$$

What about $W_3$ algebra, central extension?

- Rotating solutions: Drop condition

$$\bar{a}_x \neq -a_x^T, \quad \bar{a}_t \neq a_t^T$$

Interesting since rotating Lifshitz BH solutions have not yet been found.
Generalizations: Lifshitz theories for $hs(\lambda)$

- $hs(\lambda)$ generators $V^s_m$, $s = 2, 3, 4, \ldots$, $m = -s + 1, -s + 2, \ldots s - 1$.

- Star product:

$$V^s_m \ast V^t_n = \frac{1}{2} \sum_{u=1,2,\ldots} g^{st}_{u}(m, n; \lambda) V^{s+t-u}_{m+n}$$

- Lifshitz space-time with $z = n$

$$a_x = V^2_1, \quad a_t = V^{n+1}_n$$

- Asymptotic Lifshitz theories with $z = 2$

$$a_x = V^2_1 + \mathcal{L}V^2_{-1} + \mathcal{W}V^3_{-2} + \mathcal{U}V^4_{-3} \cdots$$

$$a_t = a_x \ast a_x$$

satisfies $F = 0$ for constant $\mathcal{L}$, $\mathcal{W}$ etc.
Generalizations: Lifshitz theories for $hs(\lambda)$

- For $L(x,t), \mathcal{W}(x,t)$ evolution equation

\[
\dot{L} = \frac{8 - 2\lambda^2}{5} \mathcal{W}'
\]

\[
\dot{\mathcal{W}} = \frac{4}{3} (L^2)' + \frac{27 - 3\lambda^2}{7} U' + \frac{1}{6} L'''
\]

\[
\dot{U} = \frac{10}{3} \mathcal{W} L' - \frac{12}{5} L \mathcal{W}' + \frac{64 - 4\lambda^2}{9} V' - \frac{1}{15} \mathcal{W}''', \quad \ldots
\]

Should be related to KP hierarchy.

- Black hole ansatz

\[
a_x = V_1^2 + L V_{-1}^2 + \mathcal{W} V_{-2}^3 + U V_{-3}^4 \ldots
\]

\[
a_t = \mu_1 a_x + \mu_2 a_x * a_x
\]

Holonomy condition difficult to impose.