Physics 140a – Problem Set 3

Due date: November 1, 2000


1. Spin susceptibility of free fermions. Consider a system of free spin-1/2 fermions in a magnetic field \( \vec{B} = B\hat{z} \). The state of each particle is specified by its wavevector, \( \vec{k} \) and spin \( s^z = \pm \hbar/2 \). As a result of the magnetic field, there will be a Lorentz force, \(- (e/c) \vec{v} \times \vec{B}\), on the fermions as well as a splitting of the two spin states. In many cases, the first effect is negligible, and we can focus on the second. The Hamiltonian is then:

\[
H = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2 - \frac{g\mu_B}{\hbar} B \sum_i s_i^z
\]

where \( \mu_B = e\hbar/2mc \) is the Bohr magneton.

(a) What is the energy \( \epsilon_{k\uparrow} \) of a state with a single fermion with wavevector \( \vec{k} \) and \( s^z = \hbar/2 \)? What is \( \epsilon_{k\downarrow} \), the energy of a state with a single fermion with wavevector \( \vec{k} \) and \( s^z = -\hbar/2 \)?

(b) Let \( \mu(T = 0) = \epsilon_F \) be the Fermi energy. The Fermi wavevectors will be different for up- and down-spin fermions. What are \( k_{F\uparrow} \) and \( k_{F\downarrow} \) in terms of \( \epsilon_F \) and \( B \)?

(c) What are the up-spin and down-spin electron densities, \( n_{\uparrow} \) and \( n_{\downarrow} \), in terms of \( \epsilon_F \) and \( B \)? What is the total electron density, \( n = n_{\uparrow} + n_{\downarrow} \)? Invert this relationship (approximately) to find \( \epsilon_F \) in terms of \( n \) and \( B \) to second order in \( B \).
(d) Find the total $z$-component of the spin, $S_z = \sum_i s_i^z$, in terms of $n$ and $B$, to first order in $B$. (Hint: what is $S_z$ in terms of $n^\uparrow$ and $n^\downarrow$?)

(e) Using your result from part (d), compute the zero-temperature Pauli spin susceptibility, 
\[ \chi_{\text{spin}} = \left( \frac{\partial S_z}{\partial B} \right)_{B=0} \]

$\chi_{\text{spin}}$ is much smaller than it would be classically. This suppression of $\chi_{\text{spin}}$ by the Pauli exclusion principle is called Pauli paramagnetism.

2. We will model a 1D ionic crystal as a linear chain of alternating positive and negative ions at $x_j = j a$. The positive ions are at the even sites; the negative ions are at the odd sites. There are $N$ ions in the crystal. The interaction between two of these ions is:
\[ U_2(r) = \pm \frac{e^2}{r} + B \frac{1}{r^{12}} \]

The $+$ sign is for ions of the same charge; the $-$ sign for oppositely charged ions. The second term models the strong short-range repulsion between the atomic cores. The cohesive energy of this ionic crystal is dominated by the potential energy:

\[ U_{\text{total}} = \frac{1}{2} \sum_{i \neq j} \left( -1 \right)^{i-j} \frac{e^2}{a} \frac{1}{|i-j|} + B \frac{1}{a^{12}} \frac{1}{|i-j|^{12}} \]

\[ = \frac{N e^2}{2a} \sum_{j \neq 0} (-1)^j \frac{1}{|j|} + \frac{N B}{a^{12}} \sum_{j \neq 0} \frac{1}{|j|^{12}} \]

\[ \equiv M \frac{N e^2}{a} + \alpha \frac{N B}{a^{12}} \]

$M$ is called the Madelung constant. Compute $M$ and $\alpha$ in the limit $N \to \infty$. 