Finite-Grid Instability in Non-Uniform Grids

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Introduction

The use of a discretization grid in particle-in-cell methods is responsible for the finite-grid instability. The instability is a spurious numerical effect due to the aliasing of different Fourier modes, that are undistinguished by the computational grid but have different effects on the particles. The finite-grid instability has been studied thoroughly in uniform grids [1, 2]; a linear theory has been developed and non-linear effects have been observed in paradigmatic cases. In recent years, the finite-grid instability has been reconsidered for non-uniform grids [3, 4], as a number of codes use such kind of mesh in order to reduce the computational effort. The effect of the finite-grid instability has been shown to limit the range of variation of the grid spacing on non-uniform grids, unless methods to overcome the instability are used [3]. For example, in the simulation of 1D electromagnetic shocks using the computational code Celest1D [3], the finite-grid instability was found in the region with larger meshes, where the stability criterion was not fulfilled, while the smaller cells in the shock region fulfilled the criterion. Notwithstanding the work performed on the subject, a systematic study of the finite-grid instability in non-uniform grids had never been done. In the present communication, a theoretical analysis of the finite-grid instability in non-uniform grids is described and the findings of an extensive investigation conducted with a simple explicit code are presented.

Linear Theory

In its simplest formulation, the new theory considers a 1D computational grid made of equal macrocells of length $2\Delta x$, each of constituted of two subcells having sizes Δx_1 and Δx_2. Every cell

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is labeled with two indices, the first, g, indicating the macrocell, the second, $\alpha = 1, 2$, to distinguish between the two subcells. The charge density in each sub-cell, due to particles of charge q and mass m, can be written as

$$\rho_{g\alpha} = \frac{q}{\Delta x_\alpha} \sum_p W_\alpha \left(X_g - x_p \right) \tag{1}$$

where X_g and x_p are the centers of the cell g and of the particle p, respectively (two assignment functions W_α are necessary). The electric field acting of the particle p can be expressed as

$$E_p = \sum_g \sum_{\alpha=1}^{2} W_\alpha(X_g - x_p) \sum_{g'} \sum_{\alpha'=1}^{2} G^{\alpha\beta}_{g-g'} \rho_{g'\alpha'} \Delta x_{\alpha'} \tag{2}$$

being $G^{\alpha\beta}_{g-g'}$ the electric field generated in the sub-cell (g, α) by a unitary charge in (g', α'). By making use of the Vlasov equation to express the effect of the electric field on the density perturbation, a relationship between the Fourier components $\widehat{\rho}_\alpha(k, \omega)$ of a mode of wavenumber k evolving as $\exp(i\omega t)$ is obtained in the form

$$\widehat{\rho}_\alpha(k, \omega) = \sum_{\beta=1}^{2} M_{\alpha\beta}(k, \omega) \widehat{\rho}_\beta(k, \omega) \tag{3}$$

where $M_{\alpha\beta}(k, \omega)$ is defined as

$$M_{\alpha\beta}(k, \omega) = \frac{(q/H)^2 \Delta x_\beta}{m \, \Delta x_\alpha} \sum_{n=-\infty}^{+\infty} \sum_{\gamma=1}^{2} \widetilde{W}_\alpha(k_n) \widetilde{W}_\gamma(-k_n) \widehat{G}^{\gamma\beta}(k) \int_{-\infty}^{+\infty} \frac{df_0/dv}{i(\omega - k_n v)} dv \tag{4}$$

being $f_0(v)$ the unperturbed velocity distribution, having indicated with $\tilde{}$ and with $\hat{}$ the Fourier transform and the Fourier series, respectively, and having set $k_n = k - \pi n / \Delta x$. For a given k, the complex frequency ω can be calculated by solving the equation

$$\left(M_{11}(k, \omega) - 1 \right) \left(M_{22}(k, \omega) - 1 \right) - M_{12}(k, \omega) M_{21}(k, \omega) = 0 \tag{5}$$

The theory sketched here represents a generalization of the classical theory of the finite-grid instability [1, 2] and can be extended to more complicated non-uniform grids.

Numerical Experiments

To perform numerical experiments of the finite-grid instability in non uniform grids, a simple explicit PIC code based on the momentum-conserving scheme [1, 2] has been implemented. The computational grid has a periodic structure with two different alternating cell sizes Δx_1 and Δx_2.

For the gathering and scattering phases of the computational cycle, the same procedure used in Celest1D [5] is employed: the computational mesh is mapped on a logical uniform grid where the CIC interpolation function is used. The PIC code used here differs from Celest1D in two respects: first, it uses a simpler explicit leap-frog algorithm and, secondly, it is based on the momentum-conserving scheme rather than the energy-conserving scheme.

The code has been used to investigate the non-linear growth of the finite-grid instability. A drifting cold beam is loaded initially in a neutralizing background. An initial perturbation $\delta x_p = C \sin(2\pi x_p/L)$ is introduced in each particle positions x_p, being L the length of the system and C a constant. A real cold beam would be stable and host stable plasma oscillations when perturbed; instead, PIC simulations show a rapidly growing mode, due to the finite-grid instability.

Figure 1 shows the evolution of the thermal energy of an initially cold beam drifting at a speed $v_0 = \omega_p \, \Delta x$. In the figure, different runs are summarized: the thermal energy is shown in gray-scale as a function of time along the y-axis; the x-axis reports different runs, each corresponding to a different value of the disuniformity parameter $\alpha = (\Delta x_1 - \Delta x_2) \, / \, (\Delta x_1 + \Delta x_2)$, while the average cell length is maintained constant. The optimal choice, corresponding to the least increase in thermal energy, is for non-uniform grids with $\alpha = 0.7$. The evolution of the thermal velocity for a uniform grid and for the "optimal" non-uniform grid is shown in detail in Fig. 2. This conclusion is not at all universal; in fact, at lower beam speeds the uniform grid ($\alpha = 0$) is optimal, and the stabilization effect of non-uniform grids is absent. In both cases, a sufficiently small value of Δt have been chosen ($\omega_{pe} \Delta t \ll 1$), in order to minimize the self-heating [2], which, otherwise, would be superimposed to the increase of temperature due to the finite-grid instability, partially hiding its effect. The results obtained with the PIC code are in agreement with those obtained by means of the linear theory presented in the previous section.

In some conditions, non-uniform grids can stabilize the finite-grid instability, compared to a uniform grid with same average grid size. This conclusion validates previous practices of using cells larger than the stability limit in some regions of the computational domain [3, 4, 6]. However, further study is required before a stability criterion of practical use can be derived for non-uniform grids.

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