Homework Problems

1. Your Aunt Gertrude bought you a camera for your birthday. While this was very sweet, the camera she bought unfortunately takes photos at uniformly random times. As a physicist, though, you think how you can take this flaw and make it fun! Your plan is to throw a ball vertically from the ground with velocity v_0. You use the camera to take photos at random times while the ball is moving upward. Your idea of fun is to determine the probability that the camera takes photos when the ball is at a given height or with a given velocity.

   (a) Find the probability distribution of the time when the camera takes photos, from when you release the ball to when it reaches its highest point.

   (b) Find the probability distribution of the velocity of the ball when the camera takes a photo.

   (c) Find the probability distribution of the position of the ball when the camera takes a photo.

2. For the ground state of the infinite square well, determine the probability distribution of the momentum of the particle, from the position-space wave function:

$$\psi(x) = \sqrt{2}\sin(\pi x)\,, \qquad x \in [0,1]\,. \tag{63}$$

   Here, x is a dimensionless position; that is, just set $\hbar = 1$. What is the maximum velocity that the particle can have quantum mechanically?

3. Find the cumulative distributions for the following probability distributions:

   (a) $p(x) = 2x$, $x \in [0,1]$

   (b) $p(x) = \delta(x - 1/2)$, $x \in [0,1]$

   (c) $p(x) = 2\Theta(x - 1/2)$, $x \in [0,1]$

   (d) $p(x) = (1+a)x^a$, $x \in [0,1]$, $a > -1$

   (e) $p(x) = -2a\frac{\log x}{x}e^{-a\log^2 x}$, $x \in [0,\infty)$, $a > 0$

   Invert the cumulative distributions for each of these cases.

4. In lecture, we discussed how to sample multi-dimensional probability distributions. Here, we will explore this some more.

(a) Recall that a 2D distribution can be written as a product of two 1D distributions using conditional probabilities:

$$p(x, y) = p(y)p(x|y)\,. \tag{64}$$

Express the 3D distribution $p(x, y, z)$ as a product of 1D distributions.

(b) Express the general n-dimensional distribution $p(\{x_i\}) = p(x_1, x_2, \ldots, x_n)$ as a product of 1D distributions.

(c) For the following 2D distributions, determine the conditional probability, $p(x|y)$.

i. $p(x, y) = -\frac{\log(xy)}{2}$, $(x, y) \in [0, 1]$

ii. $p(x, y) = \frac{(1+a)(1+b)}{2+a+b}(x^a + y^b)$, $(x, y) \in [0, 1]$.

5. Given an n-dimensional probability distribution $p(x_1, x_2, \ldots, x_n)$, we can determine the distribution of an arbitrary function of the variables $x_1, x_2, \ldots, x_n$. In this problem, we will consider the two-dimensional case, though it is easy to generalize to arbitrary numbers of dimensions.

Consider the two-dimensional probability distribution $p(x, y)$. If you want to determine the probability distribution of the function $\tau = g(x, y)$ we can evaluate

$$p(\tau) = \int dx\, dy\, p(x, y)\, \delta(\tau - g(x, y))\,, \tag{65}$$

where $\delta(x)$ is the Dirac delta function and the integral extends over the entire range of x and y.

(a) Prove that

$$\int dx\, \delta(\tau - g(x)) = \int dx\, \frac{1}{\left|\frac{dg}{dx}\right|} \delta\left(x - g^{-1}(\tau)\right)\,. \tag{66}$$

Hint: make a change of variables in the integral.

(b) For $p(x, y) = 1$ with $(x, y) \in [0, 1]$, calculate the probability distribution of the sum $\tau = x + y$ and the product $\tau = xy$.