Introductory Mechanics

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November 17, 2023

Taught at Reed College, Fall 2019, 2020

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Preface

These lectures were developed from the class I taught at Reed College during fall semesters of 2019 and 2020. Because of global circumstances this course was taught once in-person and once completely remote. For remote teaching, I pre-recorded my lectures and all corresponding demonstrations and the videos are available here:
`https://youtube.com/playlist?list=PLv1fJStSLclTqCZ8bJZPrgPnNk9gjy89G`.

Chapter 2

Introduction

Why can we trust our memories? This has a lot more to do with physics than you might think. What do we mean by "trusting memories"? We mean that our previous experiences can be used to inform future situations. This means that what we learned in the past must be applicable to the future; that is, there is a continuity through time of our experiences. A hot stove you touched yesterday hurt, therefore you know that if you touch a hot stove tomorrow it will also hurt. We can make this more physically precise by stating that experiences exhibit a **time-translation symmetry**. This means that our learned experiences are always the same (a symmetry) throughout translating or moving through time. This is obviously extremely important for conscious beings like us, otherwise we could never learn.

Even more grand a statement that follows from this is that the laws of physics do not change in time. Now, I don't mean that individual objects to not change in time; I mean that the way and rules for how objects interact with one another are always the same. For example, the rules of *Monopoly* are always the same, but any given game can have different outcomes. If the laws of physics do not change in time (they exhibit a time-translation symmetry), there ought be a concrete quantity whose value is unchanged, or **conserved**, in time. This is **energy**: that the laws of physics do not depend on time means that energy is conserved, and vice-versa. This relationship between a symmetry and a conservation law is called **Noether's theorem**, after Emmy Noether, a German mathematician.[1]

Noether's theorem is perhaps the most important result in all of theoretical physics and provides extremely strong constraints on the interactions of objects. However, depending on the system you are studying, energy may or may not be conserved. We only believe that

[1]E. Noether, "Invariant Variation Problems," Gott. Nachr. **1918**, 235-257 (1918) [arXiv:physics/0503066 [physics]].

energy is conserved for the entire universe, the only truly **closed system** we can imagine. The energy of an object can change if **work** is done on that object. Work is necessarily a concept that is outside of the object or system that you are studying. Because you can't go outside the universe, no work can be done on it and so energy is conserved.

However, not only do the laws of physics not depend on time, but they don't depend on where you are or how you are oriented. That the laws of physics are independent of your position means that they exhibit a spatial translation symmetry. Just like with time translations, Noether's theorem states that there is a conserved quantity: **momentum**. Momentum only changes if a **force** acts on your system or object. Further, the laws of physics don't depend on your orientation: throwing a ball to the north or to the west exhibits the exact same phenomena. Thus we say that physics is **rotationally-invariant**: everything (i.e., the laws of physics) are the same if you rotated the system. For rotations, Noether's theorem tells us that the corresponding conservation law is **angular momentum**. Angular momentum, a measure of an object's rotation about a fixed axis, can only change if there is a **torque** on an object.

These three conservation laws, energy, momentum, and angular momentum, will be the central components of this course. We will describe systems under which they are conserved, and use that to our advantage when making predictions for future behavior given current data. We will also discuss how work, forces, and torques break conservation laws for **open systems** (systems that interact with an external environment). Fortunately and powerfully, this breaking of conservation is not arbitrary, and we will construct powerful relationships fitting it all together.

Though this class and topics are often referred to as "classical mechanics," connoting "classical" in the Greco-Roman sense, the physics you learn this semester underlies all phenomena that we know. Conservation laws are the way that modern particle physics is formulated, and so these ideas are used throughout my own research. Though it may seem pedestrian or even pedantic at times, there is an amazingly rich structure lurking just beneath the surface. This semester, I'm thrilled to be your guide exploring Nature from this profound perspective.

2.1 Dimensional Analysis

Physics, especially introductory courses, is a problem-solving science. You have some hypothesis or question and you want to know the answer or if Nature works as you expect it to.

As a professional, card-carrying physicist, how do I know that my solution to a problem is correct? In general, I don't, but if other people independently check it, I gain confidence in its veracity. However, there are numerous tricks that a physicist has in the bag that we carry everywhere to check if a potential answer is wrong or can't possibly be correct. Throughout the semester, I'll let you in on the secrets of the trade and today we'll introduce the most powerful of all of them: **dimensional analysis**.

What makes dimensional analysis so powerful is noting that everything we can possibly measure has specific units. In this course (and much of physics), we use the SI unit system in which measured quantities are expressed in terms of the fundamental length (meter), time (second), and mass (kilogram). Every quantity we will discuss in this class is some combination of these basic units.

For example, let's say that you want to determine your speed in running 100 meters. Speed is the amount of distance you travel per unit time. We will often write speed as the letter v and we can denote its units by writing

$$[v] = \frac{\text{meters}}{\text{time}} \,. \tag{2.1}$$

Given that you ran 100 meters, to determine your speed all you need to do is to divide by the time it took to run it. If you are very fast, it might take you 10 seconds to run 100 meters. Therefore, your speed would be

$$v = \frac{100}{10}\frac{\text{meters}}{\text{seconds}} = 10 \text{ m/s} \,. \tag{2.2}$$

We can convert this into units you may be more familiar with. Let's express m/s in miles/hour. We do this by multiplying by "1" in particular ways. For example,

$$1 \text{ meter} = 1 \text{ meter} \cdot \frac{1 \text{ mile}}{1 \text{ mile}} = \frac{1 \text{ meter}}{1 \text{ mile}} \cdot 1 \text{ mile} = \frac{1}{1609} \cdot 1 \text{ mile}. \tag{2.3}$$

The factor 1/1609 is (approximately) the number of miles that you can cram into a meter. This is (much!) less than 1 because a mile is longer than a meter.

What about seconds? Let's convert 1 second into an hour:

$$1 \text{ second} = 1 \text{ second}\frac{1 \text{ hour}}{1 \text{ hour}} = \frac{1 \text{ second}}{1 \text{ hour}} \cdot 1 \text{ hour} = \frac{1}{3600} \cdot 1 \text{ hour} \,. \tag{2.4}$$

There are 3600 seconds in an hour. Now, we can convert meters/second to miles/hour as

$$\frac{\text{meters}}{\text{second}} = \frac{1}{1609} \cdot 1 \text{ mile} \cdot \frac{3600}{1 \text{ hour}} = 2.2 \frac{\text{miles}}{\text{hour}} \,. \tag{2.5}$$

So, if you ran 100 meters in 10 seconds, your speed was 10 m/s or about 22 mph.

This is an example of how dimensional analysis works. We first identify the quantity we want to calculate; in this case, a speed. Speed has units of distance over time, so to evaluate the speed we need to measure both a distance (say, 100 meters) and a time (say, 10 seconds). Dimensional analysis is the trick/shortcut/tool to get an estimate of the answer simply by ensuring that the expression has the right units (dimension). Thus, dimensional analysis is very helpful for telling you that an answer cannot possibly be correct. If the units of an expression are not correct then the answer cannot be correct.

Example

Let's see how this works in an example. Say we want to determine a distance d. We are given/measure quantities of time t, velocity or speed v, acceleration a (unit of m/s^2), and a mass m. Which of the following cannot be true?

(a) $d = at$

(b) $d = mvt$

(c) $d = v/t$

(d) $d = v^2/a$

What did you find? Note that the quantity at has units of m/s, not a distance. Also, mvt has mass in it, which is not a distance. v/t has units of m/s^2, which is acceleration. So, all of (a), (b), and (c) cannot be the expression for distance. (d) on the other hand does have the correct units:

$$\frac{v^2}{a} = \left(\frac{\text{meters}}{\text{seconds}}\right)^2 \frac{\text{seconds}^2}{\text{meters}} = \text{meters} \,. \tag{2.6}$$

We'll revisit dimensional analysis as an important tool throughout the semester.

2.2 Kinematics

For now, let's get into understanding and modeling physical systems. Physics is the study of how and why objects in the natural world change in time. We'll start this topic, of how to model objects changing, small and expand our purview in the course of the semester.

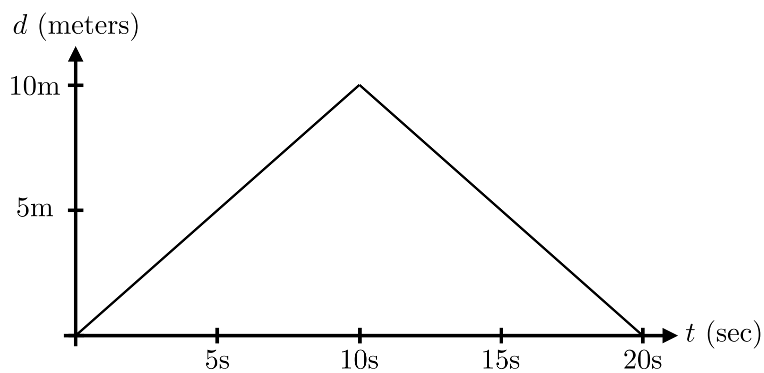
2.2.1 Displacement

So what's the simplest way that an object can change in time? I'm exhibiting it right now: an object can keep its shape and substance the same and just move through space. We live in a universe with three spatial dimensions, so ultimately we need to know how to understand motion in many dimensions. With our principle of starting small, let's ignore two of the dimensions; that is, we are just considering motion along a line. For many systems, this is very reasonable. For example, a train moves in one direction along linear tracks, so we can (often) ignore lateral motion to the tracks or motion up and down. One-dimensional motion is what I am doing at the front of class: just pacing back and forth.

If we want to be quantitative and model my motion as a function of time, there are a few things we need to address. First, how do we measure this motion? We already agreed that we use SI units so we measure the distance I travel in meters and the time over which I travel in seconds.

The next thing we need to do is to determine when to start the time and from where to measure distances. In a race, both of these things are unambiguous: the clock starts with the gun, and distances are measured from the starting line. For my pacing, it's less clear when to do either. This is not an accident: in the natural world, there is no "preferred" initial time or position. We have to impose both to be able to meaningfully speak about some physical process.

With that in mind, let's attempt to model my pacing from one side of the well to the other, and back. We will start the clock when I leave one side and also measure distance from where I start. With this agreement, we can draw a graph of my position in the well as a function of time:



Here, I have assumed that the well is 10 meters across and it takes me 20 seconds to walk out and return. This plot shows that as I just set out I am getting farther from the origin;

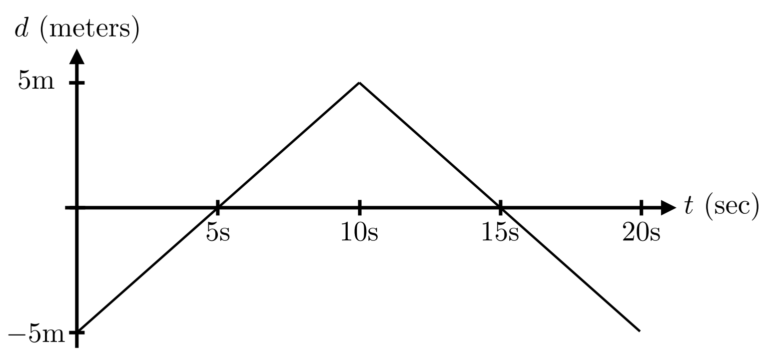
my distance is increasing. At 10 seconds, the increasing switches to decreasing distance, meaning I am getting closer to the origin as time increases; I have turned around.

Note also that I have represented distances with positive numbers (e.g., 5 meters). It is easy to represent the direction of my relationship to the origin: just use positive or negative numbers. Distances that account for relative relationships of location to an origin are **displacements**. We'll agree that positive displacements are to your right, while negative displacements are to the left of the origin.

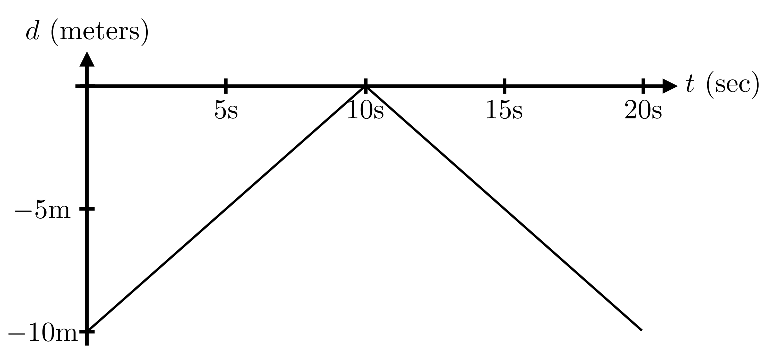
Example

With that in mind, I have another question! Consider again my walk across and back in the well. Three graphs of my motion are shown below. Can you determine where the spatial origin is for each?

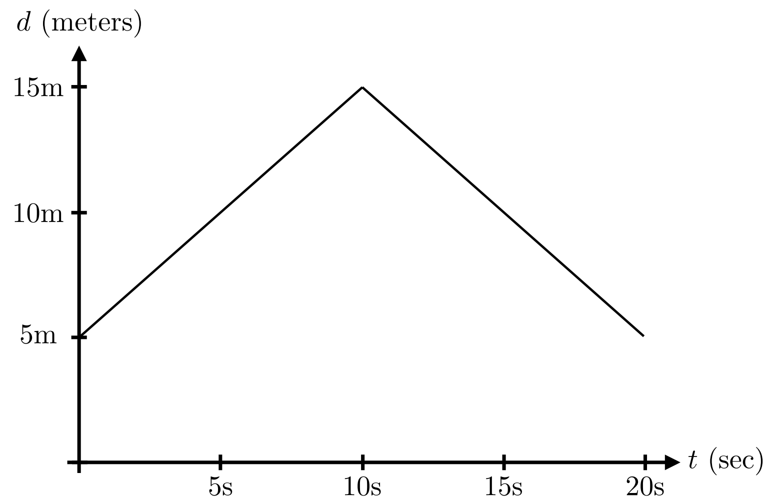
(a)



(b)



(c)



For (a), I have moved the origin to the center of the well. I start to the left of center (origin), move past it, then turn around. For (b), the origin is now at the opposite wall, and I only get to position 0 at my turning point. For (c), I've moved the origin 5 meters to the left of where I start walking! I guess it would be near the Blue Bridge.

We're about done with this, but I want to emphasize something extremely important. I've drawn four different graphs to represent me walking from one side of the room to the other. What I did was identical, we just represented it in several ways. Our particular way to describe the natural world is arbitrary and irrelevant; it does not affect what is actually happening. This is important to remember: drawing figures and graphs is very powerful for gaining understanding, but we have to remember that the drawing represents Nature, and not vice-versa.

2.2.2 Velocity

The story that the original graph tells is that I walked away from the origin for 10 seconds traveling 10 m and then returned to the origin, which took another 10 seconds. Not only does this graph tell the story of my position as a function of time, but it also tells the story of the change in my position as a function of time. That is, from one moment in time to the next, this graph encodes the rate at which my position changed. Whenevery you use the word "rate", you should think "slope", so we can identify the slope at any point in the time to study the rate of change of position.

For a function $d(t)$, the displacement as a function of time, the slope between two points at time $t + \Delta t$ and t is

$$\text{slope} = \frac{d(t + \Delta t) - d(t)}{\Delta t} \,. \tag{2.7}$$

Strictly speaking, a slope only makes sense for a line; however, we can imagine taking Δt as small as possible to determine the slope from two neighboring times, infinitesimally close to one another. This limiting procedure produces a derivative

$$\lim_{\Delta t \to 0} \frac{d(t + \Delta t) - d(t)}{\Delta t} \equiv \frac{d}{dt}\, d(t) \,. \tag{2.8}$$

We call the time derivative of displacement the **velocity** $v(t)$, which is the instantaneous change in displacement.

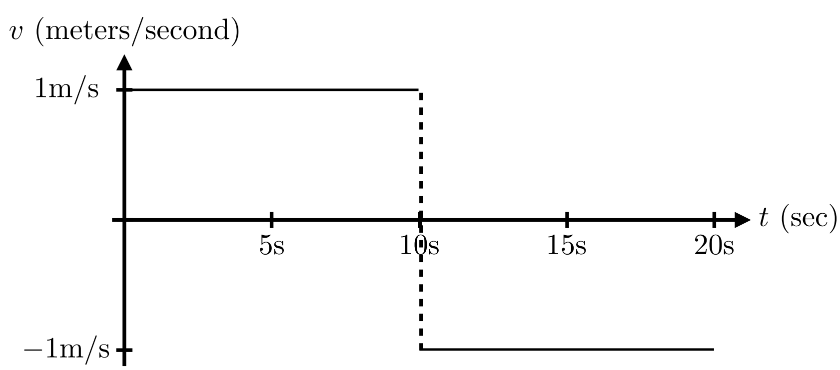
For the position versus time graph we're studying, we can produce the velocity versus time graph straightaway. In the first 10 seconds, I traveled 10 meters, so the velocity is

$$v(t < 10 \text{ s}) = \frac{10 \text{ m}}{10 \text{ s}} = 1 \text{ m/s} \,. \tag{2.9}$$

In the next 10 seconds, I traveled -10 meters (I went left instead of right), so the velocity is

$$v(10 \text{ s} < t < 20 \text{ s}) = \frac{-10 \text{ m}}{10 \text{ s}} = -1 \text{ m/s} \,. \tag{2.10}$$

The velocity versus time graph is thus



2.2.3 Acceleration

Why stop at velocity? We can also study the rate of change of velocity, called the **acceleration** $a(t)$,

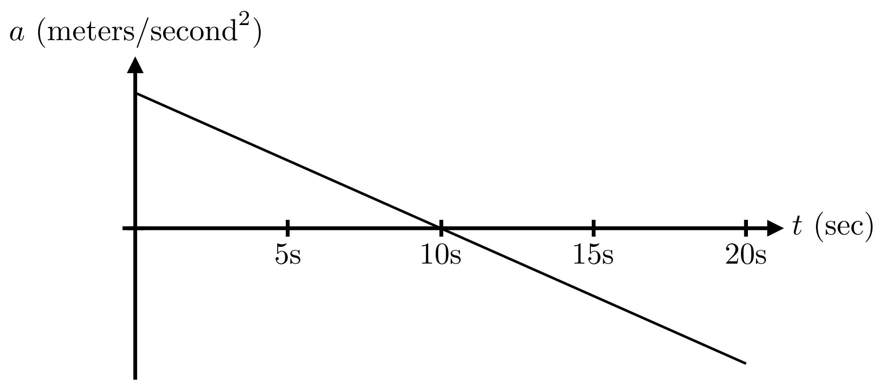
$$a(t) \equiv \frac{d}{dt}\, v(t) = \frac{d^2}{dt^2}\, d(t)\,. \tag{2.11}$$

As acceleration is the time derivative of velocity, which is itself the derivative of displacement, acceleration is the second time derivative of displacement.

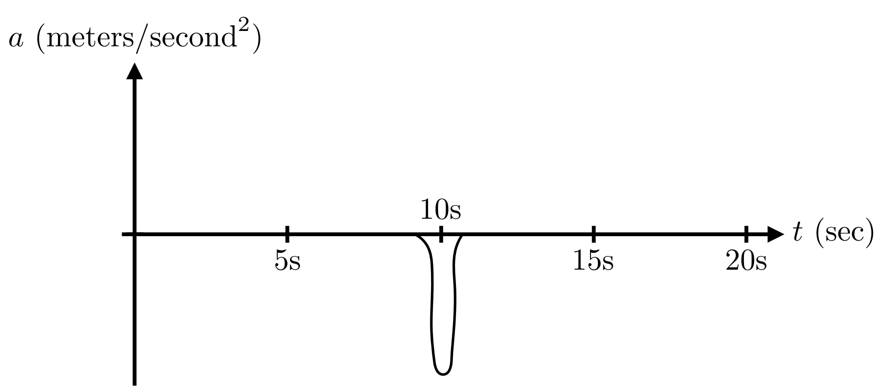
Example

Can we determine the acceleration versus time graph for my path? Below are three possible plots. Which one do you think is correct, based on the velocity versus time graph?

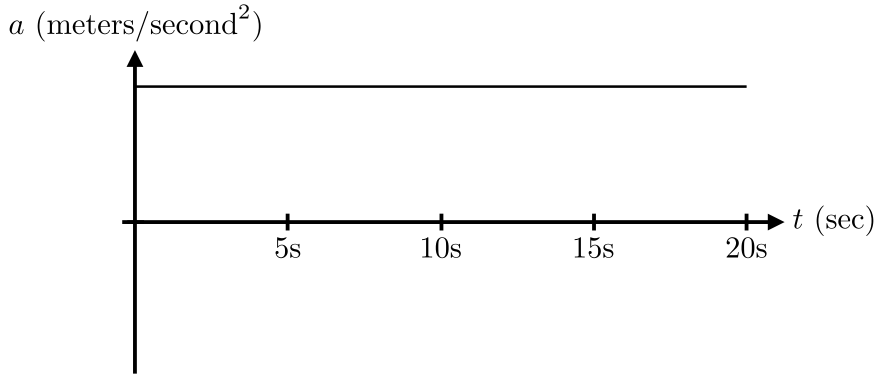
(a)



(b)

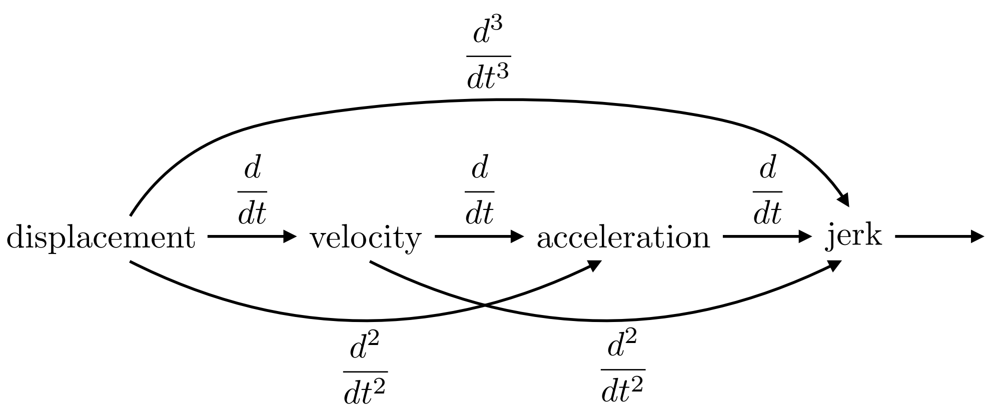


(c)



I've left off labels/ticks on the ordinate (y-axis). Note that for $t < 10$ s, the velocity is constant; it does not change in time. Therefore, $a(t < 10\text{s}) = 0$. Further, for $t > 10$s, the velocity is also constant, though perhaps a different value than the velocity before 10 s. Nevertheless, the acceleration is the instantaneous change in velocity and so too is $a(10\text{ s} < t < 20\text{ s}) = 0$. Around 10 s, however, something interesting happens: my velocity rather rapidly changes from positive to negative. Thus, around the instant of 10 s, I experience a large negative acceleration. Why negative? Because my velocity just after 10 seconds is smaller than my velocity just before. These considerations imply that (b) is the correct graph.

So, we can describe motion in different capacities by taking more derivatives



etc. In future chapters, we will make this more precise, but it's good to think about it now. Here's a question: can you "feel" velocity or acceleration? Imagine that you are on a train traveling at constant speed on very smooth rails. If you never looked out the window, could you tell that you were actually moving at all? Now, instead imagine that the conductor slammed on the brakes, violently changing the velocity of the train. Without looking out a window, could you tell that the train was stopping?

This exercise manifests a few things. First, this is referred to as a **gedankenexperiment** or thought experiment in German. Thought experiments, in which we use our experience to test our physical intuition, is an extremely powerful tool for making sense of Nature. Second, the fact that you apparently could not tell that the train was moving at a constant velocity but could tell that it was accelerating suggests a symmetry of Nature. We could all be traveling at a constant rate, and there is no experiment we could do to determine if it were so. By Noether's theorem, this constant velocity or "boost" symmetry suggests there is a conservation law. What do you think this might be? Why we feel accelerations and not velocities is encapsulated in Newton's laws, which we will discuss in coming chapters.

For now, let's just study a system which undergoes constant acceleration (note that 0 is also a constant). Let's call this constant a, which is some number. As acceleration is the rate of change in time of the rate of change in time of displacement, the dimensions/units of a are meters per second per second (m/s^2). As acceleration is the rate of change of velocity and is constant, velocity must be linear in time. In general, we can then write

$$v(t) = at + v_0 \, , \tag{2.12}$$

where $v_0 = v(t = 0)$. Note that this expression is only true for constant acceleration a. What is the displacement with constant velocity? We have the relationship

$$v(t) = \frac{d}{dt}\, d(t) \, , \tag{2.13}$$

for displacement $d(t)$. To solve for $d(t)$, we need to "undo" the derivative, or anti-differentiate. By the Fundamental Theorem of Calculus, the anti-derivative is just the integral. So, given velocity $v(t)$, we just integrate to find $d(t)$, where

$$d(t) = \frac{1}{2}at^2 + v_0 t + d_0 \, , \tag{2.14}$$

where $d_0 = d(t = 0)$. Let's now differentiate to see if this makes sense. Recall that the derivative of a power is

$$\frac{d}{dt}t^n = nt^{n-1} \, , \tag{2.15}$$

for some n. Then,

$$\frac{d}{dt}d(t) = \frac{2}{2}at + v_0 + 0 \cdot d_0 = at + v_0 = v(t) \, , \tag{2.16}$$

as expected.

For the rest of this section, we are going to use these results to measure the height of the ceiling of this lecture hall. What we will do is the following. We will throw a ball up to the ceiling, just to the point of touching the ceiling. Because the ball has an initial, non-zero velocity, and stops moving at some point, the ball is accelerating. This acceleration is due to the gravitational pull of the Earth on the ball and near the surface of the Earth is approximately constant with the value

$$a_g \equiv g = 9.8 \text{ m/s}^2 \,. \tag{2.17}$$

Note that as the ball moves up, its velocity decreases, therefore it experiences negative acceleration. Then, the displacement of the ball from the floor can be expressed as

$$h(t) = -\frac{g}{2}t^2 + v_0 t + h_0 \,, \tag{2.18}$$

where h_0 is the initial height of your hand right when you throw the ball. To know the ceiling height, we apparently need to know the time t at which the ball touches the ceiling. What else happens at that time? The ball's velocity immediately before was moving upward (positive) while immediately after is moving downward (negative). So what must the velocity be at the moment it touches the ceiling, its highest point? Zero!

With the expression for velocity,

$$v(t) = -gt + v_0 \,, \tag{2.19}$$

we know that at time $t = T_{\text{ceiling}}$, the velocity is 0:

$$v(t = T_{\text{ceiling}}) = 0 = -gT_{\text{ceiling}} + v_0 \,, \tag{2.20}$$

or, solving for initial velocity v_0, we find

$$v_0 = gT_{\text{ceiling}} \,. \tag{2.21}$$

Now, the height at $t = T_{\text{ceiling}}$ of the ball is just the ceiling height, so we can express the ceiling height as

$$h_{\text{ceiling}} = h(t = T_{\text{ceiling}}) = -\frac{1}{2}gT_{\text{ceiling}}^2 + gT_{\text{ceiling}}^2 + h_0 = \frac{1}{2}gT_{\text{ceiling}}^2 + h_0 \,. \tag{2.22}$$

So all we need to measure the ceiling height is to measure the time it takes for the ball to reach the ceiling, T_{ceiling}.

2.3 Order-of-Magnitude Estimation

We will now move on to introduce another extremely powerful tool in a physicist's toolkit: **order-of-magnitude estimation**. Along with dimensional analysis, order-of-magnitude estimation can produce profound physical insights from very simple considerations. Many very famous physics papers are in essence nothing more than dimensional analysis and estimation.

This order-of-magnitude estimation, like dimensional analysis, is not designed to produce exact results, but rather informs you of an answer within a factor of 10 or so. This might seem like it's not very useful as a factor of 10 can be a lot, but this can be very helpful for determining if your answer is reasonable and expected. Especially in homework problems, with significant amounts of algebra, first knowing the ballpark of what the solution should be is extremely insightful.

Additionally, order-of-magnitude estimation can be used to determine surprising results that may initially seem ill-defined or even impossible to solve. As such, they are often called "Fermi Problems", from Enrico Fermi, a mid-20th century physicist who mastered the art of estimation. Fermi was a scientist on the Manhattan Project and witnessed the first atomic bomb at the Trinity site. Though he and the other viewers were a few miles away from the blast, wind from the shockwave reached the viewers. Fermi tore up a piece of paper and dropped it, in doing so estimating the wind's speed. From the wind speed and the distance from the blast site, Fermi was able to estimate the yield of the bomb; that is, the total energy that it released. Though his estimation procedure was crude, he was within a factor of 2 of the correct result for the yield.

Another such problem attributed to Fermi was his question of how many piano tuners are there in Chicago? Fermi was a professor at the University of Chicago and a piano player, so this was a relevant problem. However, on its surface, it seems wildly ill-defined. The key to this order-of-magnitude estimation is that we be systematic and reasonable with all of our guesses. If we want to determine the number of piano tuners, we need to determine the number of pianos, and the number of people who might own those pianos. So, let's do this estimate.

Chicago has about 10 million people in the metropolitan area. If a house has a piano, it most likely only has a single piano. How many houses are there in Chicago? Probably there are 2 people on average in a house, so there are about 5 million houses. It is definitely too much to say that every house has a piano and too little for every 1000th house, but perhaps we could believe one piano per square block. That's about 1 piano per 10 or so houses. With 5 million houses and one piano per block, that's about 500,000 pianos. A piano is tuned how often? Every day is way too often and every 10 years is much too infrequently. If you play piano regularly, you would probably want it tuned once a year. So we need 500,000 pianos tuned once a year. How many piano tuners are needed to do this work? Well how long does it take to tune a piano? Definitely more than 1 minute, but less than a full workday otherwise business would be slow. So perhaps it takes an hour to tune a piano. One tuner could tune 8 pianos in a work day or 40 in one work week. Working 50 weeks a year, a tuner could tune 2000 pianos. If one tuner can tune 2000 pianos in a year, then 250 piano tuners could tune all 500,000 pianos in Chicago. Thus we estimate that Chicago has 250 piano tuners. In 2009, Chicago had 290 piano tuners. We are amazingly close!

While this is somewhat of a silly exercise but nevertheless exhibits the power of this tool, order-of-magnitude estimation is extremely powerful for the informed citizen. These techniques can be used to determine if a claim in the news is believable or, pardon me, bullshit. A couple of years ago, two professors at the University of Washington created a course titled "Calling Bullshit" as an exploration of tools often used in science but applied to all sorts of problems, questions, and claims. That course was further developed into a book and more information can be found at their website, `https://callingbullshit.org`.

Let's apply this estimation technique to some physics problems you may encounter. As with dimensional analysis, this can be extremely powerful for checking if your answer makes sense at all.

Example

So here's our question: what is the farthest possible distance that a human can hit a golf ball on Earth? This problem satisfies the first requirement: it is relatively ill-defined. However, by breaking it apart we can make progress and come to a solution. Let's first consider the relevant quantities we need. A golf ball is a projectile and the time that is can be in the air depends on the acceleration due to gravity, g. If g is larger, then there is greater acceleration and the ball is in the air for less time (and conversely). This also suggests that if we know the time-of-flight, we can estimate the distance. What's a reasonable time-of-flight? I'm not sure, but let's do some guesses to see what makes sense.

Is a time-of-flight of one second reasonable? That seems too short. For order-of-magnitude estimations, we would next guess 10 second time-of-flight. That does seem okay, but just to make sure, let's consider multiplying by another factor of 10. Is a 100 second time-of-flight reasonable? A full minute and a half? That seems exceptionally long (If you don't think so, then tonight sit quietly staring at the wall for 100 seconds!). So, we'll estimate a time-of-flight of 10 seconds.

Now, given $g = 10$ m/s^2 and $t = 10$ s, how do we make that a distance using dimensional analysis? Well, the units of $d = gt^2$ are a distance in meters. So we estimate, $d_{\max} \approx 10 \cdot 10^2$ m $= 1000$ m. According to the internet, the longest drive was about 500 yards, or about 500 m. So we are within a factor of 2 with simple guesses!

Example, Redux

Let's attack this problem in a different way. g is always a relevant quantity, but let's instead use the initial ball speed v_0 to estimate the longest drive. With v_0 having dimensions of meters/second, the quantity $d = v_0^2/g$ has units of length (meters). So, we just need to estimate v_0 and we can estimate d.

Let's see what makes sense. We'll think about speeds v_0 in miles per hour as that is likely what most of you are more familiar with. At the end, we'll convert to meters/second to get our result. Does $v_0 = 1$ mph make sense? No, this is a slow walking pace. What about 10 mph? Still too slow; this is a brisk jogging pace, but golf balls can go much faster. What about 100 mph? This seems reasonable; the fastest thrown baseballs in the Major Leagues are 100 mph. To see if this makes sense, let's check $v_0 = 1000$ mph. This is the speed of a fighter jet and is significantly above the speed of sound! Do golf balls break the sound barrier? Eh, no, there aren't sonic booms at the British Open. So 100 mph it is.

Dividing 100 mph by that factor of 2.2 corresponds to about 40 m/s. With this estimate for the golf ball speed, the estimate of the distance traveled is

$$d_{\max} \approx \frac{v_0^2}{g} \approx \frac{1600}{10} \text{ m} \approx 160 \text{ m} \,. \tag{2.23}$$

This is about a factor of 3 smaller than the longest drive. Again, reasonable within an order of magnitude, and with a better approximation for v_0, we can get a better estimate for $d_{\max}$.

By the way, the longest golf ball drive ever wasn't on Earth at all. Alan Shepard, an astronaut on Apollo 14, hit three golf balls on the Moon, believe to be the longest drives ever. Can we make sense of this from our calculations above? As we will discuss later in this course, the acceleration due to gravity on the Moon is less than that on Earth, both due to the facts that the Moon is smaller than Earth and because it is less dense. Anyway, we have $g_{\text{Earth}} > g_{\text{Moon}}$. A golfer on the Moon would swing a club with the same (or approximately the same) velocity as on Earth, as that is determined by the golfer's fitness. So, we expect the speed v_0 of the ball to be the same on Earth as on the Moon. Then, the distance that said golfer would hit the ball on the Moon is

$$d_{\text{Moon}} \approx \frac{v_0^2}{g_{\text{Moon}}} > \frac{v_0^2}{g_{\text{Earth}}} \,. \tag{2.24}$$

In fact, $g_{\text{Moon}} \approx g_{\text{Earth}}/6$, so the golfer could hit the ball about 6 times farther on the Moon than on Earth!

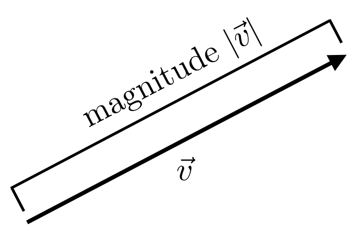
Chapter 3

Vectors

We've discussed in some detail kinematics and motion in one dimension, but this will only get us so far in understanding the phenomena of our universe. In particular, to some approximation, motion of a system may be able to be modeled as one-dimensional, but our universe is three-dimensional so we can do much more than just move along a line. So, our task now is to introduce the language used to describe objects and systems in multiple dimensions. Our fundamental object for doing so will be a **vector**.

3.1 Representation of Vectors

Simply, a vector is a quantity that has a magnitude and a definite direction in space. A magnitude is a non-negative number that specifies the length of the vector. The direction of a vector is simply how the vector points with respect to a specified origin. For example, we might draw the vector $\vec{v}$ as:



The length of the arrow is the vector $\vec{v}$'s magnitude, denoted as $|\vec{v}|$. The head of the arrow tells us what direction the vector points, with respect to the origin which, by convention, is located at the tail of the arrow.

We've already seen vectors in this course before, but not really in that language. In the previous chapter, we had modeled my strolling across the well. We had defined one point in the well to be the origin, and my displacement from that origin could be plotted as a function of time. This displacement $\vec{d}$ is a vector; it has a magnitude and direction. The magnitude $|\vec{d}|$ is the distance from the origin; that is, the number of meters, say, that a measuring device (stick, tape, etc.) would read extended from me to the origin. We had also discussed a direction: displacement is positive to the right and negative to the left. In this way a displacement $\vec{d}$ of $\vec{d} = -5$ m means or can be read as "five meters to the left of the origin."

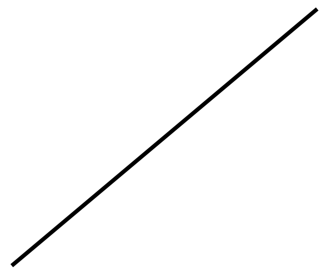
Vectors require a well-orderedness to be well-defined and unambiguous. In the example just discussed, this essentially follows from the well-orderedness of the real numbers. A negative real number means left of the origin (0) while a positive real number means to the right.

Enough about one dimension, though. Our universe has three spatial dimensions: left-right, up-down, forward-back. What we mean by a "dimension" can be understood practically and just the number of ways that one can travel through space. Mathematically, a dimension is an independent direction in space. That is, one can move left or right completely independently of up and down. One can move left, say, without ever changing the relative value of "up-ness".

Three dimensions is hard to draw, so let's just work in two dimensions which will essentially illustrate all of the subtleties of multiple dimensions. This paper is two-dimensional: one can move left-right or up-down on it. This can be done completely independently. For example, the line

only lives in the left-right dimension. By contrast, the line

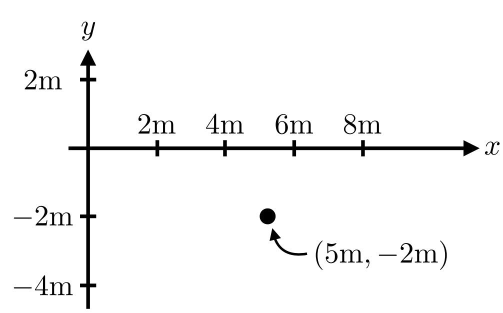
only exists in the up-down dimension. These two dimensions/lines are independent in the sense of the following. For the left-right line, we can move along the line, changing our left-right location, but without changing the up-down location at all. The following line



exists in both the left-right and up-down dimensions. Traveling along it changes both the left-right and up-down positions.

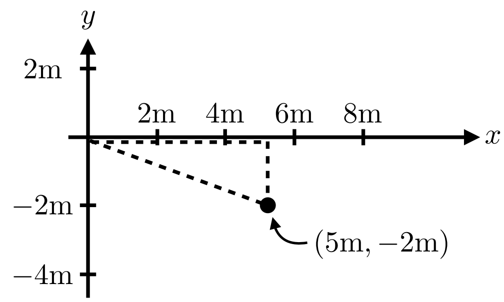
How can we represent these features? As always, we first need to pick an origin; a point from which everything is measured. We'll denote it by a dot

From this dot, we then need to denote our two dimensions. We can do this by drawing axes, both left-right and up-down. The left-right axis is called the abscissa and the up-down axis is the ordinate, but we often colloquially just say "x-axis" and "y-axis", respectively. Now, we can represent any point on the paper by its left-right coordinate ("x-component") and up-down coordinate ("y-component"). For example, a point with $x = 5$ m and $y = -2$ m would be somewhere like so



We can also express this point as an ordered pair of numbers (x_0, y_0), which are the x- and y-coordinates of the point with respect to the origin, respectively.

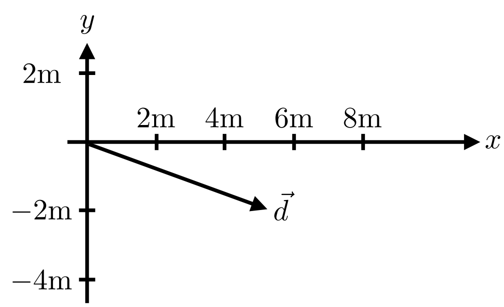
This picture is very powerful. We can immediately determine the distance of this point from the origin. For the example given, we can draw the triangle:



This triangle has sides of length 5 m and 2 m that meet at a right angle. Therefore, the distance to the origin, d, the hypotenuse of the triangle, is simply the application of the Pythagorean theorem:

$$d = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}\,. \tag{3.1}$$

Now, if this point represented my location traveling in the well, we can define an arrow that points from the origin to my position; my displacement vector $\vec{d}$. We can draw this as



This vector can be expressed as

$$\vec{d} = (5\text{ m}, -2\text{ m}) = (5\,\hat{\imath} - 2\,\hat{\jmath})\text{ m}\,. \tag{3.2}$$

Here, $\hat{\imath}$ and $\hat{\jmath}$ are called **unit vectors** and simply represent the direction of the axes in our drawings.

Example

The power of good notation is that it immediately enables an extension of what we've developed here. For example, given my initial position of $\vec{d} = (5\text{ m}, -2\text{ m})$, what is my position vector after walking in the y-direction 10 m? The possible answers are:

(a) $\vec{d}_{\text{new}} = (15 \text{ m}, -2 \text{ m})$

(b) $\vec{d}_{\text{new}} = (5 \text{ m}, 8 \text{ m})$

(c) $\vec{d}_{\text{new}} = (5 \text{ m}, -2 \text{ m})$

(d) $\vec{d}_{\text{new}} = (5 \text{ m}, 10 \text{ m})$

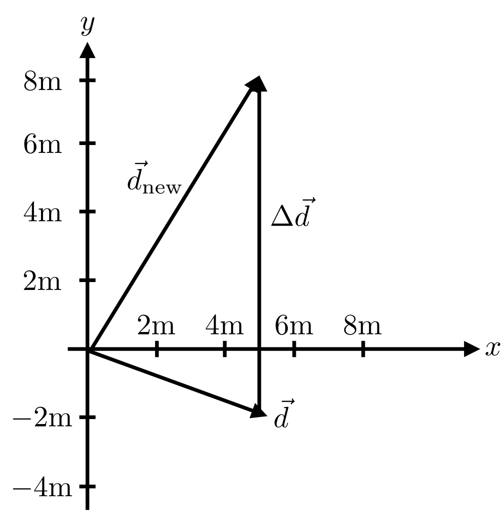
To solve this problem, we can use **vector addition**, which exploits the independence of the dimensions. My initial vector is $\vec{d} = (5 \text{ m}, -2 \text{ m})$, as measured from the origin. Now, from this point, I move 10 m in the y-direction. That is, if I consider my current location as the origin, then my displacement from this point $\Delta\vec{d}$ is

$$\Delta\vec{d} = (0, 10 \text{ m})\,. \tag{3.3}$$

Now, here's the beauty of all of this. To find my displacement $\vec{d}_{\text{new}}$ from the "true" origin, all I have to do is all $\vec{d}$ to $\Delta\vec{d}$:

$$\vec{d}_{\text{new}} = \vec{d} + \Delta\vec{d} = (5 + 0 \text{ m}, -2 + 10 \text{ m}) = (5 \text{ m}, 8 \text{ m})\,. \tag{3.4}$$

Note that addition proceeds component-wise. This addition has a lovely picture, too. First, my current displacement is measured from the origin. Next, I consider my current position the origin, and then displace $\Delta\vec{d}$ from it. That is, we draw the vector $\Delta\vec{d}$ starting from the head of $\vec{d}$



Finally, the total displacement $\vec{d}_{\text{new}}$ is the vector formed from connecting the "true" origin to the end of $\Delta\vec{d}$. This "head-to-tail" visual vector addition is extremely powerful and we'll exploit it throughout the course. By the way, the magnitude of my new displacement is

$$|\vec{d}_{\text{new}}| = \sqrt{5^2 + 8^2} = \sqrt{25 + 64} = \sqrt{89}\,. \tag{3.5}$$

Sorry, these aren't nice numbers!

A few things to note: independent dimensions represent distinct orthogonal/perpendicular/right directions in which one can travel in space. Our first step in basically every physics problem we will encounter is to identify the origin and set up orthogonal axes, as physics in the different dimensions will largely be independent of one another.

This notation also allows us to express lines or general curves in two dimensional space. For example, the equation for a line is the standard

$$y = mx + b\,. \tag{3.6}$$

A generic point p on the line with x- and y-coordinates $p = (x, y)$ can be expressed as

$$p = (x, mx + b)\,. \tag{3.7}$$

That is, given a point x, the point y is uniquely determined by the equation for a line. Further, the line is one-dimensional; you only need to specify a single point (x) to know both coordinates.

This can be extended to any curve. For example, a parabola can be expressed as

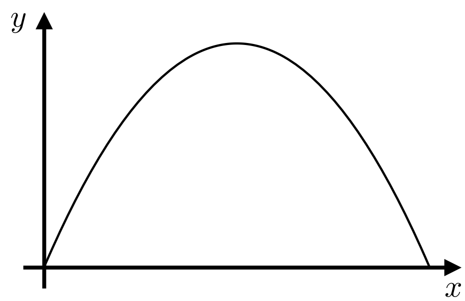
$$y = ax^2 + bx + c\,, \tag{3.8}$$

for some real numbers a, b, c. Does this parabola represent a one-dimensional object? Why or why not? Can you construct curves for which *every* point in two-dimensional space lives somewhere on the curve?

3.2 Projectile Motion

Using this new language, we will first study the physics of **projectile motion**. A projectile is an object that flies freely, only influenced by the effects of gravity. More colloquially, a projectile can be a thrown baseball, a golf ball, the *Vomit Comet* airplane, or a whale that has unfortunately materialized high in the atmosphere. We start studying projectiles because the physics it manifests is quite simple. As we discussed previously, a dimension is defined as an independent direction in space. This property of independence implies that for studying physics in multiple dimensions, we can consider the properties and physics of each dimension separately and then combine the analysis at the end. Projectile motion exhibits sufficiently interesting, yet simple, phenomena in each dimension, which is why it is a good place to start.

First, let's motivate the physics of projectiles. We'll be studying physics applicable to our everyday, human-sized experience. On this scale, the surface of the Earth is, to very, very good approximation, flat. This direction along the ground will be one of our dimensions. The other dimension we consider is vertical: projectiles (like a baseball) travel both along the ground direction and up (and down) through the air. So, the two dimensions that we study can nicely be drawn on a page or blackboard. For example, the trajectory of a ball that we throw across the softball field might look like



Here, x is the distance along the ground and y is the height of the ball and you throw it to the right from the origin.

Let's identify the physics in each of these dimensions. First, the vertical dimension. Again, I want to emphasize that because vertical and horizontal are independent dimensions, we can analyze them separately. From our everyday experience, gravity acts exclusively in the vertical direction. To good approximation (so far, anyway), gravity enacts a constant acceleration of g (≈ 10 m/s^2) toward the ground. We had already introduced the formula for position as a function of time for an object undergoing constant acceleration of magnitude g, the height y as a function of time is

$$y(t) = -\frac{g}{2}t^2 + v_{0y}t + y_0 \,, \tag{3.9}$$

where v_{0y} and y_0 are the initial vertical velocity and height, respectively. Here, "initial" means at time $t = 0$. Note that I am careful to denote the velocity as the initial y-component of velocity, v_{0y}, as that is all that is relevant for the formula for height.

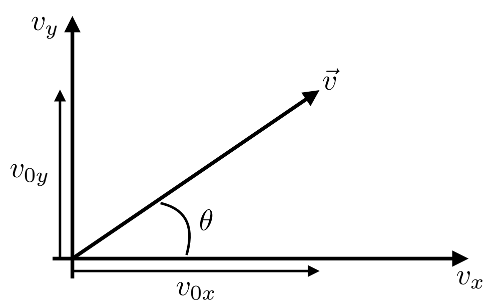
For horizontal motion, we want a corresponding formula for $x(t)$. Is there any acceleration horizontally? That is, if I drop a ball, does it spontaneously accelerate along the direction of the ground? Nope, or at least I don't think so! So, if there is no acceleration ($a = 0$) in the x-direction, the expression for the horizontal position as a function of time is simple:

$$x(t) = v_{0x}t + x_0\,, \tag{3.10}$$

where v_{0x} is the initial (time $t = 0$) velocity in the x-direction. So, we have the two equations for horizontal and vertical position:

$$x(t) = v_{0x}t + x_0\,, \qquad\qquad y(t) = -\frac{g}{2}t^2 + v_{0y}t + y_0\,. \tag{3.11}$$

For the initial velocities v_{0x} and v_{0y}, note that these are simply the x- and y-components of the two-dimensional initial velocity vector $\vec{v}_0$. This can be expressed and illustrated as



$$\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath} = (v_{0x}, v_{0y})\,. \tag{3.12}$$

This vector can equivalently be expressed in terms of the magnitude of $\vec{v}_0$, v_0, and the angle θ above the horizontal as

$$v_{0x} = v_0 \cos\theta\,, \qquad\qquad v_{0y} = v_0 \sin\theta\,. \tag{3.13}$$

That is, our system of equations reads

$$x(t) = v_0 \cos\theta\, t + x_0\,, \qquad\qquad y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta\, t + y_0\,. \tag{3.14}$$

By taking derivatives, we can find velocities and accelerations in these two dimensions.

Again, I want to emphasize a consequence of the independence of the horizontal and vertical dimensions. Every object, regardless of its velocity vector initially, accelerates with g by gravity (ignoring air resistance). This result prompted a famous example of astronaut on the Moon (where there is no air resistance) to demonstrate that a feather and a hammer would hit the Moon's surface at the same time when dropped from the same height.

Example

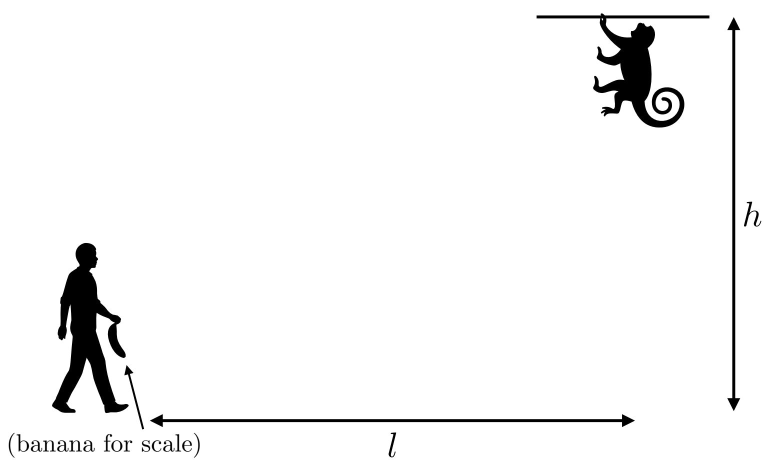
This also segues into an example we'll study for much of the rest of this section. Here's the setup. A monkey is hanging from a branch a distance l from you horizontally and a height h above you. The monkey has a very loudly growling stomach so you think you can give it one of your bananas. This is a very smart (and hungry!) monkey so it doesn't like things thrown at it. In defense, the monkey releases its grip on the branch at the exact moment that anyone throws something at it. To ensure that the banana hits the monkey, where should you aim when you throw it?

(a) Above the monkey's original location

(b) At the monkey's original location

(c) Below the monkey's original location

To answer this question, let's first draw a picture:



Let's next write down the kinematic equations for the banana. With you located at the spatial origin and throwing the banana at time $t = 0$, the trajectory of the banana is

$$x_b(t) = v_0 \cos\theta\, t\,, \qquad y_b(t) = -\frac{g}{2}t^2 + v_0 \sin\theta\, t\,. \tag{3.15}$$

We assume that you throw the banana with speed v_0 at an angle θ above horizontal. Now, the monkey just drops itself from the branch at time $t = 0$, so its horizontal position is constant in time: $x_m(t) = l$. Its height, on the other hand, is

$$y_m(t) = -\frac{g}{2}t^2 + h\,, \tag{3.16}$$

noting that the monkey's initial velocity is 0: it drops from rest.

Now, if the banana hits the monkey at some time T, this means that the horizontal and vertical components of the positions of the monkey and banana are identical. For horizontal positions, this enforces that

$$x_b(T) = x_m(T) = v_0 \cos\theta \, T = l \,, \qquad \Rightarrow \qquad T = \frac{l}{v_0 \cos\theta} \,. \tag{3.17}$$

Now, plugging this into the equations for the heights of the monkey and banana and setting them equal, we have

$$y_b(T) = y_m(T) \quad \Rightarrow \quad -\frac{g}{2}\left(\frac{l}{v_0\cos\theta}\right)^2 + v_0 \sin\theta \, \frac{l}{v_0 \sin\theta} = -\frac{g}{2}\left(\frac{l}{v_0\cos\theta}\right)^2 + h \,. \tag{3.18}$$

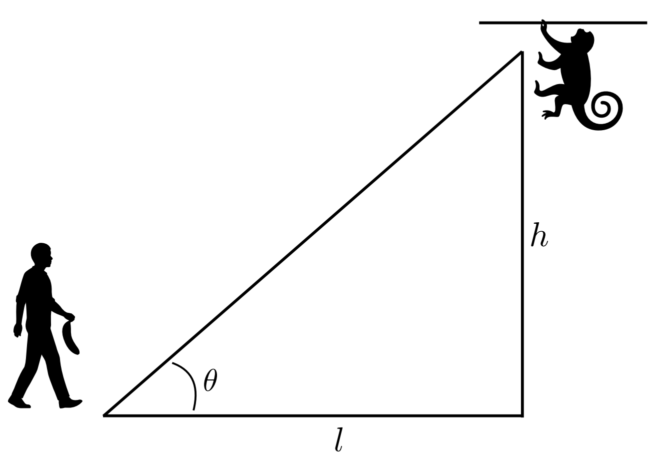
Massaging this expression we find that the terms proportional to g cancel each other. The way to interpret this is that the banana and the monkey "fall" for the exact same distance. Canceling these terms, we find that the banana hits the monkey if

$$\frac{v_0 l \sin\theta}{v_0 \cos\theta} = h \,, \tag{3.19}$$

or that

$$\tan\theta = \frac{h}{l} \,. \tag{3.20}$$

Remember, θ is the angle above the horizontal of the initial velocity. $\tan\theta$ is the ratio of the height to the length of a right triangle and h/l is exactly the ratio of the height and length of the triangle formed from you, the monkey and the ground:



That is, you should aim right at the monkey!

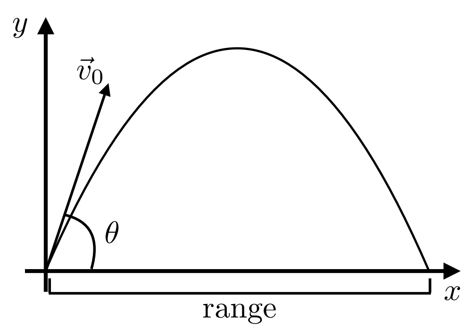
Just our luck, we can test this out today! Before we do, I want to tell you a couple of quick stories. First there's the adage that if it wiggles, it's biology. If it stinks, it's chemistry. If it's dirty, it's geology. And if it doesn't work, it's physics!

Also, I'm a theoretical particle physicist, which means that I stay far away from experiment. There's a phenomena called the "Pauli effect" named after the theorist Wolfgang Pauli, who is perhaps most famous for the Pauli exclusion principle. Pauli was notorious for destroying any and all experimental apparatuses he came in contact with. A statement of the Pauli effect is that Pauli and a working experiment can not be found in the same room. Apparently, this effect was extremely strong. Once an experiment in Goettingen, Germany, failed for some unknown reason, completely randomly. The experimentalists were baffled, until they learned that at almost the exact time of the failure, Pauli was changing trains in . . . Goettingen!

Hopefully I have better luck than Pauli! (See `https://youtu.be/m54v6gfDWN8`)

3.2.1 The Range Formula

Finally, I want to present an estimate and plausibility for the **range formula**. That is, given level ground and an initial velocity v_0 an angle θ above the horizontal, how far does a projectile travel? The picture is



The range is a distance, so we can use dimensional analysis to determine its dependence on the given quantities.

The only dimensionful quantities in the problem are the initial speed v_0 and acceleration g. The range r is formed by some product of these quantities raised to powers a and b:

$$r = v_0^a g^b \,. \tag{3.21}$$

We can determine a and b by matching units. The units of r are meters m and the units of

$v_0^a g^b$ are

$$[v_0^a g^b] = \frac{\mathrm{m}^a}{\mathrm{s}^a}\frac{\mathrm{m}^b}{\mathrm{s}^{2b}} = \mathrm{m}^{a+b}\mathrm{s}^{-a-2b}\,. \tag{3.22}$$

If this is to reduce to just meters, we know that the second units must be eliminated, or that $-a-2b=0$. Further, a single distance unit means that $a+b=1$. These two equations can be solved by first adding them together:

$$(-a-2b)+(a+b) = -b = 0+1 = 1\,, \tag{3.23}$$

or that $b=-1$. It then follows that $a=2$. That is, the range is proportional to

$$r \propto \frac{v_0^2}{g}\,. \tag{3.24}$$

What about dependence on the angle θ? Well, we would have to solve and reorganize the kinematic equations, but we can note two limits. First, if $\theta = 0°$, the projectile is shot parallel to the ground from the ground, so has 0 range. Also, if $\theta = 90°$, the projectile travels straight up vertically and lands where it started. Again, this is 0 range. These considerations suggest that r is proportional to

$$r \propto \frac{v_0^2 \sin(2\theta)}{g}\,. \tag{3.25}$$

When $\theta = 0°$, $\sin 0° = 0$ and when $\theta = 90°$, $\sin(2\cdot 90°) = \sin 180° = 0$. We need a bit more work to justify it, but the range formula is actually just this:

$$r = \frac{v_0^2 \sin(2\theta)}{g}\,. \tag{3.26}$$

3.3 Relative Velocity

One of my favorite pastimes as a kid on long car trips was staring at the other cars on the road and watching them pass by. It's a mesmerizing thing: your car could be traveling 70 miles per hour with respect to the ground, but the car next to you could appear to be at rest. Or, if your parent(s) had a particularly heavy lead foot, the cars next to you could appear to be traveling backward, even though everyone was moving forward. Looking across the road, to oncoming traffic, they would zip by at huge speeds, much faster, it would appear, than we were traveling individually. How do we make sense of these relative velocities and speeds? We can exploit vectors and their formalism to study the problem.

Let's analyze this problem, starting in one dimension. I'm imagining that I'm in the back seat of my parents' car, on the long drive to my grandparents' house. While driving through the deserts of the western US, there's a lot of time to think about physics, so I'm wondering how fast the cars in the other lanes appear out my window. Let's say that the velocity of my car is $\vec{v}_{\text{me}}$, while the velocity of the truck with Florida plates is $\vec{v}_{\text{truck}}$. Hold on a second; we have a bit more to define. Just as we needed to specify an origin from which to measure distances when I walked across the well, we need to define with respect to what we measure velocities.

Velocity is defined as a difference of displacement over change in time:

$$\vec{v}(t) = \frac{d}{dt}\,\vec{d}(t)\,, \tag{3.27}$$

so velocity is independent of the spatial origin we choose. A different spatial origin just corresponds to some displacement of our position by some constant vector $\vec{d}_0$:

$$\vec{d}(t) \to \vec{d}(t) + \vec{d}_0\,. \tag{3.28}$$

Because $\vec{d}_0$ is independent of time, velocity is unaffected:

$$\vec{v}(t) \to \frac{d}{dt}\left(\vec{d}(t) + \vec{d}_0\right) = \frac{d}{dt}\vec{d}(t) = \vec{v}(t)\,. \tag{3.29}$$

However, we had previously done the gedankenexperiment in which we considered sitting in a very smooth train traveling at a constant velocity. Recall that we couldn't tell if the train was actually moving, if our eyes were closed or if we weren't looking out the window.

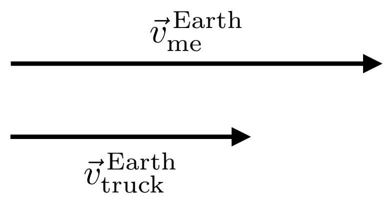
An interpretation of this is that there is nothing special about a particular *absolute* constant velocity. All that matters are relative velocities. So, whenever we talk about a velocity, we need to specify with respect to *what*. Those objects that have 0 velocity with respect to one another are said to define a **frame of reference**, or simply just **frame**.

So, when we say that we have velocity $\vec{v}_{\text{me}}$, we need to define the frame in which this velocity is defined. Typically, it's the velocity with respect to the Earth, so for analyzing the problem at hand, we denote our velocity and the velocity of the truck from the frame of the Earth as

$$\vec{v}_{\text{me}}^{\,\text{Earth}}\,,\, \vec{v}_{\text{truck}}^{\,\text{Earth}}\,. \tag{3.30}$$

Using these quantities, can we determine the velocity of the truck I would see out my window? That is, what is the velocity of the truck in my reference frame, $\vec{v}_{\text{truck}}^{\,\text{me}}$?

Let's draw some vectors to express this setup:

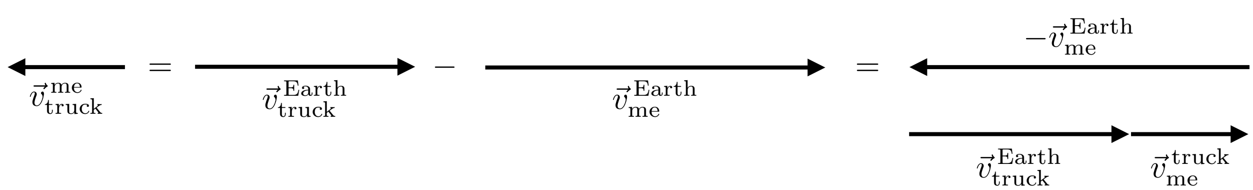


From these, we want to determine $\vec{v}_{\text{truck}}^{\,\text{me}}$. Let's think about how we get into my reference frame. Imagine that you're just standing on a curb and a car drives by to your right (another gedankenexperiment!). That is, in your frame, the car drives right. Another way to imagine that the car moves to the right is not that you are at rest (with respect to the Earth), but that the car is at rest and you are moving to the left! That is, to move from one frame to another, you need to subtract the relative velocity of the two frames.

That is, $\vec{v}_{\text{me}}^{\,\text{me}} = 0$ as I am at rest by definition in my frame. To determine the velocity of the truck in my frame, I just subtract my velocity with respect to Earth from the truck's velocity to the Earth:

$$\vec{v}_{\text{truck}}^{\,\text{me}} = \vec{v}_{\text{truck}}^{\,\text{Earth}} - \vec{v}_{\text{me}}^{\,\text{Earth}}\,, \tag{3.31}$$

or, in arrow notation:



I can always determine another relative velocity from two velocities in a single frame:

$$\boxed{\vec{v}_{\text{obj}}^{\,\text{frame 2}} = \vec{v}_{\text{obj}}^{\,\text{frame 1}} - \vec{v}_{\text{frame 2}}^{\,\text{frame 1}}\,.} \tag{3.32}$$

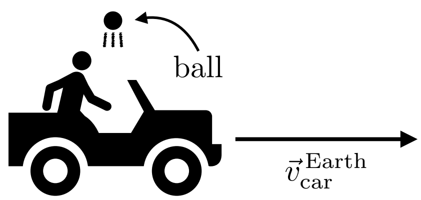
This is true as a vector equation, so it is the rule for relating two frames in any number of dimensions.

Example

Using this formalism, let's analyze the following system. Imagine that you are sitting in the backseat of a convertible, with the top down (wayfarers on...), traveling at a constant velocity with respect to the Earth. You throw a ball vertically in the car's frame. What happens? Does the ball:

(a) Land behind you (b) Land in front of you (c) Land right on you

A picture of this is:



Let's analyze this systematically. We are given the following:

$$\vec{v}_{\rm car}^{\,\rm Earth} = v_{\rm car}\hat{\imath}\,, \tag{3.33}$$

where $v_{\rm car}$ is the car's speed and we assume that it is moving along the x-axis. Also, the ball's velocity in the frame of the car is

$$\vec{v}_{\rm ball}^{\,\rm car} = (-gt + v_{\rm ball})\hat{\jmath}\,. \tag{3.34}$$

Recall, in the car's frame, the ball travels vertically, in the y-direction. The ball's initial speed in this frame is $v_{\rm ball}$, and it, of course, undergoes acceleration $-g$ due to gravity.

Now, what is the velocity of the ball with respect to the Earth? From our master relative velocity formula, we have

$$\vec{v}_{\rm ball}^{\,\rm Earth} = \vec{v}_{\rm ball}^{\,\rm car} - \vec{v}_{\rm Earth}^{\,\rm car}\,. \tag{3.35}$$

This is a bit weird, as we know $\vec{v}_{\rm car}^{\,\rm Earth}$, but need $\vec{v}_{\rm Earth}^{\,\rm car}$. No fear, these are simply opposites of one another

$$\vec{v}_{\rm Earth}^{\,\rm car} = -\vec{v}_{\rm car}^{\,\rm Earth}\,. \tag{3.36}$$

(Can you convince yourself of this?) Therefore, the velocity of the ball with respect to the

Earth is

$$\vec{v}_{\text{ball}}^{\,\text{Earth}} = \vec{v}_{\text{ball}}^{\,\text{car}} - \vec{v}_{\text{Earth}}^{\,\text{car}} = \vec{v}_{\text{ball}}^{\,\text{car}} + \vec{v}_{\text{car}}^{\,\text{Earth}} = v_{\text{car}}\hat{\imath} + (-gt + v_{\text{ball}})\hat{\jmath}\,. \tag{3.37}$$

Now, we need to determine the position of the ball after it has traveled up and come down to the vertical location of the car. To determine the displacement of the ball with respect to the Earth, we simply integrate its velocity:

$$\vec{d}_{\text{ball}}^{\,\text{Earth}}(t) = (v_{\text{car}}t)\,\hat{\imath} + \left(-\frac{g}{2}t^2 + v_{\text{ball}}t\right)\hat{\jmath}\,. \tag{3.38}$$

I have set the initial position at $t = 0$ to be the 0 vector. The ball starts going up at $t = 0$ and comes back to vertical displacement of 0 when

$$-\frac{g}{2}t^2 + v_{\text{ball}}t = 0\,, \tag{3.39}$$

or when

$$t = \frac{2v_{\text{ball}}}{g} \equiv T_{y=0}\,. \tag{3.40}$$

What is the ball's x-position at this time? We simply plug in $t = T_{y=0}$ to find

$$x_{\text{ball}}(T_{y=0}) = v_{\text{car}}T_{y=0} = \frac{2v_{\text{ball}}v_{\text{car}}}{g}\,. \tag{3.41}$$

What is the car's x-position at this time? It's identical because, by the vector nature of velocity, the car and the ball have the same x-component of velocity. That is,

$$x_{\text{car}}(T_{y=0}) = \frac{2v_{\text{ball}}v_{\text{car}}}{g}\,. \tag{3.42}$$

That is, because both the x- and y-components of the position of the ball and the car are identical at $T_{y=0}$, the ball lands in my lap!

Also, what shape would you see as the trajectory of the ball (you being at rest with the Earth)? Note the ball's x-component of position is

$$x_{\text{ball}}(t) = v_{\text{car}}t\,, \tag{3.43}$$

or that

$$t = \frac{x_{\text{ball}}}{v_{\text{car}}} \,. \tag{3.44}$$

Plugging this into the expression for the y-component of the ball's position, we find

$$y_{\text{ball}}(t) = -\frac{g}{2}t^2 + v_{\text{ball}}t = -\frac{g}{2}\frac{x_{\text{ball}}^2}{v_{\text{car}}^2} + \frac{v_{\text{ball}}x_{\text{ball}}}{v_{\text{car}}} \,. \tag{3.45}$$

That is, the height of the ball that you see as a function of its horizontal position is just a parabola:

$$y_{\text{ball}}(x_{\text{ball}}) = -\frac{g}{2}\frac{x_{\text{ball}}^2}{v_{\text{car}}^2} + \frac{v_{\text{ball}}x_{\text{ball}}}{v_{\text{car}}} \,. \tag{3.46}$$

The independence of perpendicular spatial directions is extremely profound and produces concrete predictions we can test with experiment. (See `https://youtu.be/eqQ05FYZoZU`)

3.4 Circular Motion

In this section, we are going to describe the kinematics of a very common situation that one encounters in physics. We have discussed the kinematics and description of linear motion, generalized that to multiple dimensions with vectors, and now we will discuss **circular motion**. Precisely what we will study in this section is the kinematics of traveling in a circle, but this analysis is applicable broadly to any motion that is non-linear: merry-go-rounds, a right turn in a car, a loop-the-loop rollercoaster, etc.

I want to start with a question that we'll address throughout this lecture. Imagine that you are on a merry-go-round, giggling with your friends. The merry-go-round is spinning/rotation at a constant speed. Are you accelerating? And if so, in what direction are you accelerating?

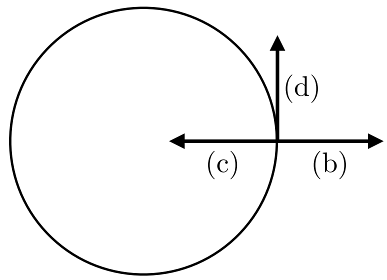
(a) No acceleration

(b) Yes, accelerating outward

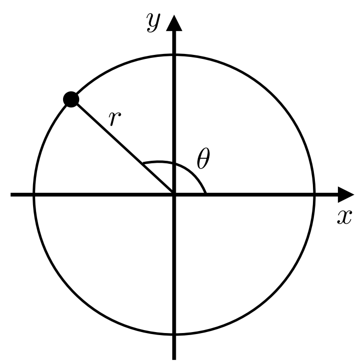
(c) Yes, accelerating inward

(d) Yes, accelerating forward

For (b), (c), and (d), the direction of acceleration of you on the merry-go-round is:



We'll come back later to answer this definitively, but now we'll set up the language to describe circular motion. Let's first identify positions on a circle, with a convenient origin:



We have drawn a circle of radius r with the origin at the center of the circle. A point/position $\vec{p}$ on the circle can be represented in Cartesian coordinates (x, y space) as

$$\vec{p} = (r\cos\theta, r\sin\theta)\,, \tag{3.47}$$

where θ is the angle as measured above the $+x$-axis, as illustrated. At this point, there is no motion; this is just a point, static, unchanging. If we move in a circle, our distance from the origin remains unchanged, but the angle θ changes in time. That is, we consider time dependence of

$$\vec{p}(t) = (r\cos\theta(t), r\sin\theta(t))\ , \tag{3.48}$$

with r a constant radius. What is the simplest temporal dependence for the angle $\theta(t)$? Just as we studied for motion in one dimension, the simplest motion is linear in time:

$$\theta(t) = \omega t + \theta_0\,. \tag{3.49}$$

We'll mostly restrict to this case in this lecture. This angular time dependence is called constant **angular velocity** motion: ω is called the angular velocity as it is the rate that

$\theta(t)$ changes in time:

$$\frac{d}{dt}\theta(t) = \omega\,. \tag{3.50}$$

θ_0 is a constant, the original location in angle of your position.

So, we'll study position that changes in time as

$$\vec{p}(t) = (r\cos(\omega t + \theta_0), r\sin(\omega t + \theta_0)) = r\cos(\omega t + \theta_0)\hat{\imath} + r\sin(\omega t + \theta_0)\hat{\jmath}\,. \tag{3.51}$$

Given this position, let's do our standard analysis of finding velocity and acceleration. Velocity is just the time derivative of position:

$$\vec{v}(t) = \frac{d}{dt}\vec{p}(t) = \frac{d}{dt}\left(r\cos(\omega t + \theta_0)\right)\hat{\imath} + \frac{d}{dt}\left(r\sin(\omega t + \theta_0)\right)\hat{\jmath}\,. \tag{3.52}$$

We need to take derivatives of cosine and sine to find velocities. To do this, we exploit the **chain rule** of derivatives. To take the derivative of a function of a function, the rule is

$$\frac{d}{dt}f\left(g(t)\right) = \frac{df}{dg}\frac{dg}{dt}\,. \tag{3.53}$$

For the x-component of velocity, we want to take the derivative

$$\frac{d}{dt}\cos(\omega t + \theta_0)\,. \tag{3.54}$$

Let's call $f(\theta) = \cos\theta$, and $\theta(t) = \omega t + \theta_0$. Then, the derivative is

$$\frac{d}{dt}\cos(\omega t + \theta_0) = \frac{d\cos\theta}{d\theta}\frac{d}{dt}(\omega t + \theta_0) = \omega\frac{d\cos\theta}{d\theta}\,. \tag{3.55}$$

So, what's the derivative of $\cos\theta$? The answer is, which I won't explain in more detail here,

$$\frac{d}{d\theta}\cos\theta = -\sin\theta\,. \tag{3.56}$$

Therefore, we have

$$\frac{d}{dt}\cos(\omega t + \theta_0) = -\omega\sin(\omega t + \theta_0)\,. \tag{3.57}$$

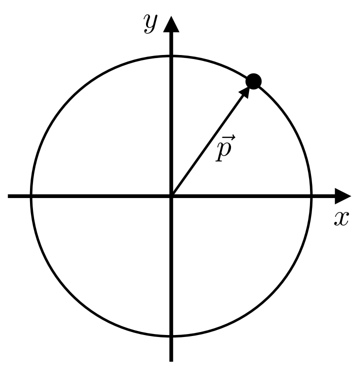
Similarly,

$$\frac{d}{dt}\sin(\omega t + \theta_0) = \omega\cos(\omega t + \theta_0)\,. \tag{3.58}$$

It then follows that the expression for velocity when moving at constant speed around a circle is

$$\vec{v}(t) = -\omega r\sin(\omega t + \theta_0)\hat{\imath} + \omega r\cos(\omega t + \theta_0)\hat{\jmath}\,. \tag{3.59}$$

It's interesting to pause here for a second and think for a bit about what this velocity is telling us. First, recall that, in the prescribed coordinate system, the position vector lies along a radial line:



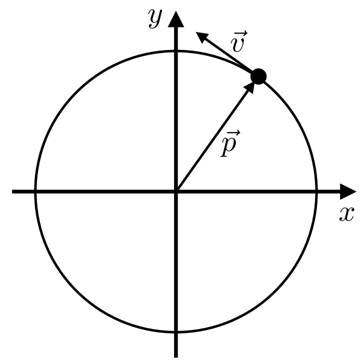
At this point on the circle, what is the velocity vector? To answer this question, let's use a trick from freshman geometry. Let's schematically denote the position vector as

$$\vec{p}(t) = a\hat{\imath} + b\hat{\jmath} = (a, b)\,. \tag{3.60}$$

a and b are the cosine and sine bits, but for this argument that is just distracting. In terms of a and b now, the velocity vector is

$$\vec{v}(t) = \omega(-b\hat{\imath} + a\hat{\jmath}) = \omega(-b, a) \propto (-b, a)\,. \tag{3.61}$$

If I have to vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (-b, a)$, what angle do they make with one another? 90°! Therefore, if we are moving counterclockwise around a circle, the velocity vector is **tangent** to the circle:



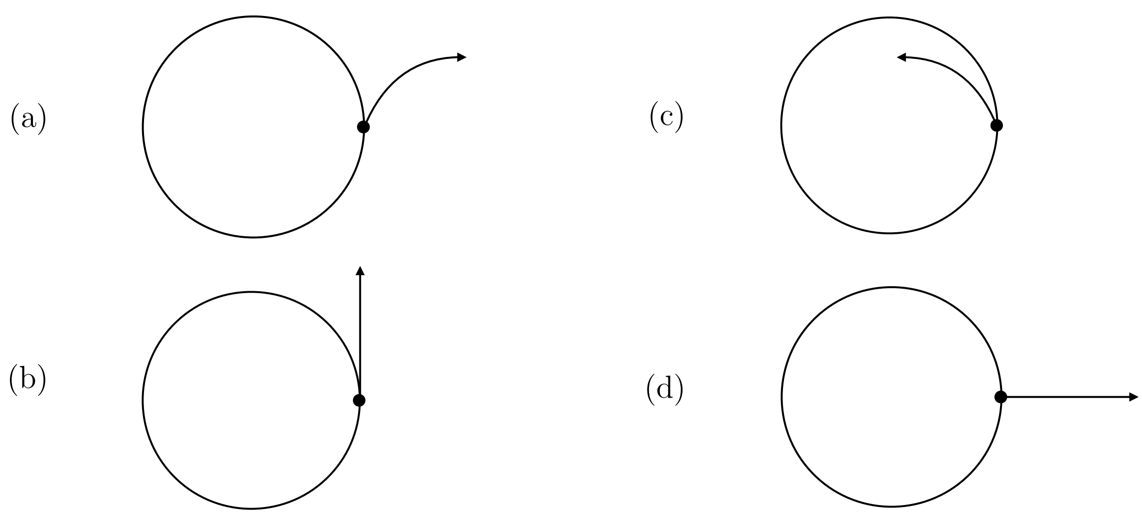
This orthogonality can also be captured in a **dot product**. The dot product of two vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (c, d)$ is defined to be

$$\vec{v}_1 \cdot \vec{v}_2 = ac + bd \,. \tag{3.62}$$

If the dot product is 0, the vectors are orthogonal. Let's see this for position and velocity:

$$\begin{aligned} \vec{p}(t) \cdot \vec{v}(t) &= (r\cos(\omega t + \theta_0), r\sin(\omega t + \theta_0)) \cdot (-\omega r\sin(\omega t + \theta_0), \omega r\cos(\omega t + \theta_0)) \qquad (3.63) \\ &= -\omega r^2 \cos(\omega t + \theta_0)\sin(\omega t + \theta_0) + \omega r^2 \cos(\omega t + \theta_0)\sin(\omega t + \theta_0) \\ &= 0 \,. \end{aligned}$$

Now, I have a question for you. Say an object is traveling at constant speed in a circle, for example, by being swung with a string. If the strong is cut, how does the ball/object travel afterward?



Also, consider a slingshot. You twirl the sling round and round and then let go. To hit a target in front of you, how and when should you release it?

Finally, let's get back to where we started, and address acceleration. If an object is moving in a circle at constant speed, it is accelerating? That is, is its velocity changing in time? We said that the speed was constant, so by definition, the magnitude of velocity is not changing. However, that's not the only way to change velocity. As a vector, changing velocity can mean changing its magnitude and/or direction. Clearly, the direction of velocity is changing as you move in a circle (otherwise you wouldn't, well, move in a circle). So, we are definitely accelerating.

We can just take another derivative of velocity to determine acceleration, but it's useful to take a step back first, and think about the direction of acceleration. To do this, let's return to our gedankenexperiment space. First, imagine that you are sitting in a car that is at rest. Now, the driver accelerates forward by pressing on the gas pedal. What do you feel? What is your response to a forward acceleration? You are pushed backward; that is, you feel a push in the opposite direction to acceleration. An analogous thing happens if the driver now brakes. Acceleration is backward (forward motion is slowing down), yet you are pushed forward. Keep this in mind.

Now, imagine that you are sitting at the edge of a merry-go-round that is being pushed at a constant rate by delinquent fourth graders. Just thinking about what you would feel, how and in what direction do you feel a push? So, from our thought experiment in the car, what is the direction of acceleration?

Let's now calculate acceleration from the derivative of velocity. We have

$$\begin{aligned}\frac{d}{dt}\vec{v}(t) = \vec{a}(t) &= \frac{d}{dt}\left(-\omega r\sin(\omega t+\theta_0), \omega r\cos(\omega t+\theta_0)\right) \\ &= \left(-\omega^2 r\sin(\omega t+\theta_0), -\omega^2 r\cos(\omega t+\theta_0)\right)\,.\end{aligned} \tag{3.64}$$

But wait! Recall that the position vector for moving around a circle was

$$\vec{p}(t) = (r\cos(\omega t+\theta_0), r\sin(\omega t+\theta_0))\,. \tag{3.65}$$

Therefore, acceleration is just $\vec{a}(t) = -\omega^2\vec{p}(t)$. As a real number, $\omega^2 > 0$ and so acceleration points in the opposite direction as the position vector. As $\vec{p}(t)$ pointed from the center of the circle to its perimeter, acceleration points from the perimeter to the center. Thus, this acceleration is "center seeking," or **centripetal acceleration**.

We will end this section with one more observation. Let's evaluate the magnitudes of velocity and acceleration and see if there's a relationship between them. Recall that the velocity vector was

$$\vec{v}(t) = (-\omega r \sin(\omega t + \theta_0), \omega r \cos(\omega t + \theta_0)) \ . \tag{3.66}$$

Its magnitude, via Pythagorus, is

$$|\vec{v}| = \sqrt{(-\omega r \sin(\omega t + \theta_0))^2 + (\omega r \cos(\omega t + \theta_0))^2} = \omega r \,, \tag{3.67}$$

because $\cos^2\theta + \sin^2\theta = 1$. As such, ω is called the angular velocity. What about acceleration? We have

$$|\vec{a}| = \sqrt{(-\omega^2 r \sin(\omega t + \theta_0))^2 + (-\omega^2 r \cos(\omega t + \theta_0))^2} = \omega^2 r \,. \tag{3.68}$$

Note that the angular velocity in terms of the (linear) speed is

$$\omega = \frac{|\vec{v}|}{r} \,. \tag{3.69}$$

Plugging this into the expression for acceleration, we have

$$|\vec{a}| = \omega^2 r = \left(\frac{|\vec{v}|}{r}\right) r = \frac{|\vec{v}|^2}{r} \,. \tag{3.70}$$

This will be a very useful result. The magnitude of centripetal acceleration is squared velocity divided by the radius of circular motion.

Chapter 4

Newton's Second Law

In this lecture, we are really going to start in earnest attempting to describe *why* physical phenomena happen. The first few lectures of the course were setting the stage, introducing the language of vectors, kinematics, position, velocity, and acceleration, but now we have a complete enough vocabulary we can use it to construct new sentences. So with that in mind, here we are going to introduce **forces** and their consequences.

4.1 Forces and Acceleration

Let's go back, as we often have, to our thought experiment of the train moving at constant velocity on a very smooth track. As mentioned many times, there is no experiment we can do on the train to determine if it is actually moving or just at rest. The laws of physics are independent of one's velocity, or, more precisely, independent of one's frame of reference. Balls fall when dropped with acceleration $-g$, water is still in a cup, etc., and this is (one) consequence of independence of the physics in different dimensions. By contrast, if the train accelerated, the driver put on the brakes to avoid colliding with cows on the tracks, you will know it. You will feel *pulled* forward dramatically. Pull is an action word and specifically what a pull does is change your motion. You were happily traveling at a constant rate, ne'er the wiser, and then the driver changed your motion/velocity by slamming on the brakes. A change in velocity is acceleration and a pull, precisely but even colloquially, is a force. That is, forces (things/actions that enact change) are responsible for acceleration.

This is a profound observation and its mathematical consequence is encapsulated in Newton's second law, which we will get to in a second. If a force imparts acceleration, then you can exert a force on your friend to make them fall over. That is, a push (= force) can change their idle standing (= at rest) to falling over (= moving). Thus, you made them accelerate by pushing them. Further, there's clearly a difference between pushing a friend and pushing an elephant. With the same exerted push, your friend will fall over, but the elephant may not even know you are there! That is, though a force is the same, the corresponding acceleration (= change of motion) can be very different. So what differs between your friend and a pachyderm that could be responsible for the different acceleration? Well, other than four legs and a trunk, an elephant is much more massive than your friend. With more mass, an equivalent push/force changes motion less.

These considerations and thought experiments motivate **Newton's second law**, which essentially encapsulates all of mechanics that we will learn in this course. Newton's second law is:

$$\boxed{\vec{F}_{\text{net}} = m\vec{a}\,.} \tag{4.1}$$

Here, m is the mass of the object of interest, $\vec{a}$ is its acceleration (a vector, recall), and $\vec{F}_{\text{net}}$ is the total or net force acting on that object. Force is a vector: you can push harder or softer and affect the magnitude of force, and you can also push in different directions. Thus Newton's second law enables you to predict the motion of an object based on the forces acting on it.

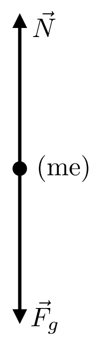
Before we make sense of this, the phrase "Newton's second law" begs the question of if there is a Newton's first law, and for that matter, how many laws are there? Is it just laws all the way down? Well, there are three "Newton's laws" and the first and third can be thought of as consequences of the second. So, we really won't specifically talk about them, like we will for the second law. (There's also a *Monty Python* joke here: "Thou shalt count to three, no more, no less. Five is right out!")

So let's see how Newton's second law works in some examples. First, and most simply, let's analyze the forces on me, just standing here. Am I accelerating? Nope, so therefore what is the net force that is acting on me? If $\vec{a} = 0$, then $\vec{F}_{\text{net}} = 0$, too! We can break this apart some more to make sense of how $\vec{F}_{\text{net}} = 0$. What forces are acting on me; that is, what individual pushes and pulls are exerted on me? First, gravity is pulling me toward the center of the Earth. This gravitational force is also called **weight** and its magnitude is just equal to my mass times our old friend, acceleration g:

$$|\vec{F}_{\text{grav}}| = mg\,. \tag{4.2}$$

What other forces are acting on me? That is, am I feeling another push? Actually, you can ask yourself this question. Simply by sitting in your seat, do you feel something pushing on you? I feel the floor pushing on my feet. In your case, you should feel the chair pushing, um, somewhere else. So there is also a force from a surface keeping us upright. Such a force is called a **normal force** because it acts normal, or perpendicular, to the surface. I can't think of or feel other forces (other than the weight of the world on my shoulders...), so this is all we have.

Let's draw a picture to represent the forces on me:



Such a picture is called a **free-body diagram** and it represents all forces acting on an object. The dot at the center is me, but I ignore all spatial extent in such a diagram. Gravity, $\vec{F}_g$, pulls me down while the floor's normal force $\vec{N}$ pushes me up. The net force, $\vec{F}_{\text{net}}$, is just the vector sum of the forces acting on me:

$$\vec{F}_{\text{net}} = \vec{N} + \vec{F}_g = (N - mg)\hat{\jmath}\,, \tag{4.3}$$

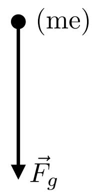
where N is the magnitude of the normal force. By Newton's second law, this is equal to my mass times acceleration, $m\vec{a}$. But what is acceleration? $\vec{a} = 0$, so $\vec{F}_{\text{net}} = 0$ which then implies that

$$N = mg\,. \tag{4.4}$$

That is, the floor pushes me up juuuuust enough to counteract the force of gravity.

4.1.1 Equivalence Principle

What if there's no floor? Not that we blow it up or something, but if we jump out of a plane (ignoring air resistance)? Now, there is no normal force acting on me, so my free-body diagram is



Newton's second law then says that

$$\vec{F}_g = (-mg)\hat{\jmath} = m\vec{a}\,, \tag{4.5}$$

or that $\vec{a} = -g\hat{\jmath}$. Of course this is consistent with that we've been playing with this whole time: every object accelerates at the same rate under the effects of gravity, independent of its mass.

I want to belabor this point a little bit. First, we are only able to make this claim if the mass that multiplies $\vec{a}$ in Newton's second law (**inertial mass**) is the same as the mass that multiplies g (**gravitational mass**). In principle, these two quantities could be different, but we have 0 experimental evidence to suggest that. The claim/axiom that inertial mass is equivalent to gravitational mass is called the **equivalence principle**, and its assumption provides a deep and profound prediction. Actually, let's this about these two systems we just discussed a bit more. When I am standing here, or you are sitting in your chair, do you actively "feel" a push? Well, yes, as argued earlier, I feel a push on my feet. However, and this is odd, while I actually feel a push, my acceleration is 0. By contrast, imagine jumping out of a plane. Ignoring air resistance, do you actually feel anything pulling you? Nope, you feel like you are floating, or are in **free fall**. Nevertheless, while you feel no pull, you are accelerating!

This is very weird and the equivalence principle suggests that gravity, as we typically think about it, is not a force in the same way that a push or pull is. Actually, gravity is not a true force at all; it is simply the manifestation of the curvature of space and time under the influence of massive objects. This is completely described by Einstein's **theory of general relativity**, but for this class, we'll just assume that gravity acts like a force, which will be good enough for our study.

Example

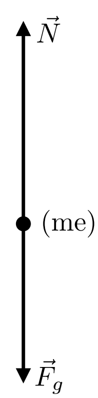
Let's go back to the topic at hand and think about another system: an accelerating elevator. Let's say that you are in an elevator that is accelerating upward by a. How does the magnitude of the normal force of the floor on you compare to your weight? Is the normal force:

(a) larger than weight? (b) same as weight? (c) less than weight?

To solve this problem, let's use Newton's second law and draw a free-body diagram. First, unless you jumped up, you are accelerating at the same rate as the elevator, a. By Newton's second law, the net force is then

$$\vec{F}_{\text{net}} = m\vec{a} = ma\,\hat{\jmath}\,, \tag{4.6}$$

where the acceleration is upward (positive). Now, to a free-body diagram. What are the forces that act on you? There's always gravity, and we would feel the normal force from the floor of the elevator pushing upward. Anything else, acting directly on you? Nope! So, the free-body diagram is



and so the net force is

$$\vec{F}_{\text{net}} = \vec{N} + \vec{F}_g = (N - mg)\hat{\jmath}\,. \tag{4.7}$$

By Newton's second law, this is supposed to equal

$$\vec{F}_{\text{net}} = (N - mg)\hat{\jmath} = m\vec{a} = ma\,\hat{\jmath}\,, \tag{4.8}$$

or that $N = m(g + a)$. Assuming that $a > 0$, the normal force must be larger than your weight, mg.

4.1.2 Newton's First Law

We will discuss more consequences of Newton's second law,

$$\vec{F}_{\text{net}} = m\vec{a} \tag{4.9}$$

shortly, and introduce a new force, **friction**, that is a major player in our everyday lives. Here, we will pause to briefly discuss Newton's first law.

The mass "m" that appears on the right side of Newton's second law is called the "inertial mass," as it is the property of an object that opposes a change in motion. For the same force, a larger mass object accelerates less than a smaller mass object. The word "inertia" shares its origin with inert, which means unskilled or inactive, from Latin. That is, "inertial mass" is a measure of the inactivity of an object. It takes more force to make a more massive object "active".

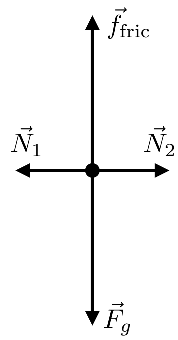
As I mentioned previously, there are three of Newton's laws, the second of which is the most general, and the first and third of which follow from the second law. So, one way forward is to simply ignore laws 1 and 3, but they are interesting in their own right and historically relevant, so I want to spend a little time thinking about them. The velocity of an object is a measure of its motion, while acceleration is a measure of an object's change of motion in time. Therefore, by the second law, to change an object's motion, a force must act on it. This is the first law:

> An object will travel with constant velocity until acted upon by an external force.

So, indeed that follows from Newton's second law.

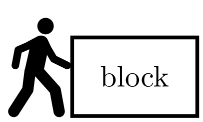
4.2 Friction

We'll visit the third law at the end of this section. Until then, let's discuss a force that is necessary for nearly all of our mundane activities. The only force we have discussed that one object can act on another object is normal force. As the name suggests, normal force acts normal or perpendicular to a surface. For example, the normal force of the floor holds me up and prevents me from falling to the center of the Earth. However, this is clearly not the only way that objects can exert forces on other objects. Right now, I am exhibiting a miracle of science: I am holding up a piece of paper between my fingers. This paper is clearly at rest, not accelerating, so Newton's second law says that its net force must be 0. Let's draw its free-body diagram:

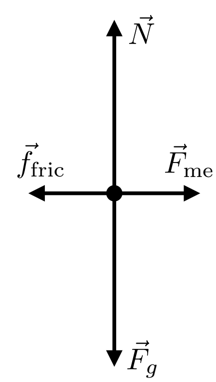


The paper has mass, so feels a force of gravity, and my fingers exert normal forces on the left and right of the paper. Note that this cannot be all that is happening. If this were all, then there would be a net force and the paper would therefore accelerate. My fingers exert another force on the paper: friction. Simply by holding the paper, my fingers stick slightly to it, in a direction tangential to the surface of my fingers. This friction arises from weak bonding at the atomic level between my fingers and the paper. However it works microscopically, we can assign it a name and a value. Friction is often denoted $\vec{f}_{\text{fric}}$, with a lowercase "f" and opposes relative motion in its reference frame. Adding friction, we then have a free-body diagram that can have 0 net force, and therefore actually describes the situation at hand.

So, what is this friction force and what are its properties? Let's consider analyzing the forces acting on this block sitting on this table. I will attempt to push it, but let's say that it remains at rest. The physical picture of this is



and free-body diagram



If the block doesn't move, $\vec{a} = 0$, and so the net force is 0. In the absence of friction, my pushing force would have accelerated the block to the right, so friction acts to oppose this motion in its frame. So, to ensure that $\vec{a} = 0$, we must have that $\vec{f}_{\text{fric}} = -\vec{F}_{\text{me}}$.

If I pushed hard enough, however, the block would move, if I overcame friction. What is this minimum force to move the block? In general, I don't know until I measure it, but we are scientists so we get to make hypotheses. Note that the normal force is always just

$$\vec{N} = -\vec{F}_g\,, \tag{4.10}$$

regardless of how hard I push. From your experience, how do you think the minimum force to push the block would change if I doubled the mass, and therefore doubled the normal force? What if the size of the block surface that touched the table changed, but the mass was unchanged? I will postulate a hypothesis, and then we will test it. My hypothesis is that the maximum friction force is proportional to the normal force:

$$|\vec{f}_{\text{fric}}| \leq \mu_s|\vec{N}|\,. \tag{4.11}$$

Here, because the block is not initially moving, the proportionality constant μ_s is called the **coefficient of static friction**. Once my push force exceeds $\mu_s|\vec{N}|$ in magnitude, then the block should move.

Example

We will test two aspects of this hypothesis here and now. (See `https://youtu.be/CaCsiLHS_-A` for the demonstrations.) The first test will be to see the response if the mass is doubled. From our free-body diagram, the magnitude of the normal force is just the weight of the block, $|\vec{N}| = mg$. Therefore, if the mass m is doubled, how should the maximum frictional force be affected?

(a) doubled (b) halved (c) unchanged (d) other

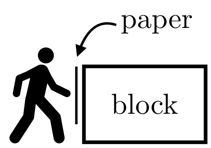
Next, this maximal friction force is solely determined by its relationship to normal force, and all other properties are encapsulated in μ_s, which is just some number. What do you predict, with our hypothesis for the form of the maximal friction force, happens if the surface area of the block that touches the table changes? We leave the weight of the block the same, but we will just rotate it on its side, so the surface that touches the table is halved. Will the maximal friction force be:

(a) doubled (b) halved (c) unchanged (d) other

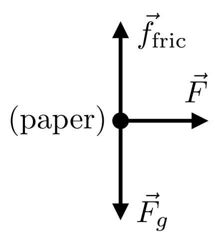
Friction is a very important feature of the world and without it, everyday experiences would be dramatically altered. I want to show a tragic clip of what happened to elementary school students when they went through a day with no friction. They were never the same afterward. (See `https://youtu.be/TcdcYBRIk3k`)

4.2.1 Newton's Third Law

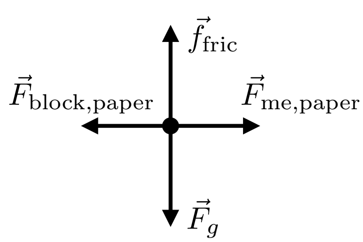
Finally, let's go back to the first thing we talked about and finish up Newton's laws. Let's consider again us pushing a block, but this time, let's slide a piece of paper between us and the block:



We are pushing with a force $\vec{F}$, but we'll say that friction is sufficient to prevent movement of the block. In particular, this also means that the paper is at rest. As it is at rest, its net force is 0. What are the forces on the paper? Clearly I am pushing on the paper. Gravity is pulling on the paper, and there is friction between me and the paper. What is the block doing? Well, let's draw the free-body diagram thus far



Friction could cancel the force of gravity on the paper, but there would still be a net horizontal force, $\vec{F}$. The paper is at rest so this is impossible. Therefore, the block must be pushing on the paper with force $-\vec{F}$:



Now, the paper was just a thought device; I can remove it and the block must still be pushing to the left with force $-\vec{F}$. That is, if I push on the block with force $\vec{F}$, the block pushes on me with force $-\vec{F}$. Newton's third law is typically stated as:

Every action has an equal and opposite reaction.

Later, we will see that this is the statement of conservation of momentum.

Example

In this example, I have a string attached to a bar, which is itself attached to a weight, off of which another string is attached. We are going to do two things: first, I will quickly yank the bottom string, and second, I will slowly pull the bottom string. So, I have two questions for you:

1. When I yank the bottom string quickly, which string will break first?

   (a) top (b) bottom (c) both at the same time

2. When I slowly pull the bottom string, which string will break first?

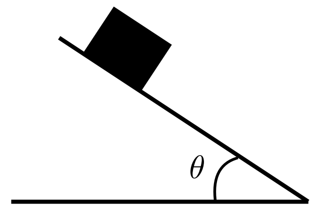
   (a) top (b) bottom (c) both at the same time

Let's think about these two cases for a second before we try it out. For the set-up in which you yank the bottom string, we are imparting a large force very quickly. Even though it is a large force, because of the large mass of the weight, its acceleration is small, which means that the acceleration of the string above it is small. Only the string at the bottom has a large acceleration and so will break first.

By contrast, if we slowly pull the bottom string, then the bottom string, weight, and the top string accelerate together. The top string additionally is pulled by the weight of the, uh, weight, so the tension on the top string is greater than that of the bottom string. With enough pull at the bottom, the top string will break first. Let's test this out! (See `https://youtu.be/Hui_LbLBqjg`)

4.2.2 Coefficient of Static Friction

We had introduced the coefficient of static friction μ_s through our hypothesis for how the force of friction acts on objects. One way to determine the value of μ_s is to just see what is the minimum force you must apply to get a block moving, and compare that force to the normal force on the block. This technique requires measuring forces accurately, which we may not be able to do easily. Instead, I want to introduce another method here that is much simpler and only requires measuring one angle. Here's the setup: I am going to put a block on an incline or ramp whose angle with respect to the horizontal I can vary, like so



Now, I tilt the ramp, increasing θ juuuuust until the block starts to move. That angle, $\theta_{\min}$, then tells me information about the coefficient of static friction, μ_s.

We'll analyze this in a second, but I want to first ask you to think about how $\theta_{\min}$ can be related to μ_s. Is μ_s equal to:

(a) $\mu_s = \cos\theta_{\min}$

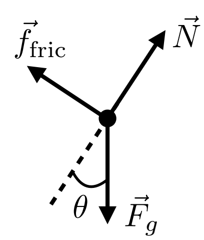
(b) $\mu_s = \sin\theta_{\min}$

(c) $\mu_s = \tan\theta_{\min}$

(d) some other relation

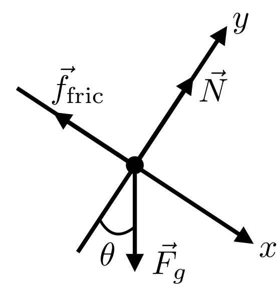
Don't fully analyze the problem now; just think about limiting cases and what each expression would imply for the value of μ_s. For example, what *should* the value of μ_s be if the minimum angle was 0, $\theta_{\min} = 0$? By contrast, what *should* the value of μ_s be if $\theta_{\min} = 90°$?

Okay, let's analyze this system now. As always in this game, we draw a free-body diagram for the block:



Note the normal force is always perpendicular to the ramp and the block would want to slide down the ramp, so the friction force acts in the direction up the ramp.

As the free-body diagram demonstrates, this is manifestly (that is, "obviously") a two-dimensional system, so we need to identify our axes appropriately, in a way to simplify the physical description. The physics cannot depend on the coordinates we use, but we can exploit properties of the system. First, our goal is to understand the force of friction, $\vec{f}$, so it might be easiest to align the friction force with an axis. Correspondingly, the normal force, $\vec{N}$, would also point along an orthogonal axis. So, we will set up coordinates as



where the x-axis points down the ramp and the y-axis is perpendicular off the ramp.

With this coordinate system, we can now write the vectors in component form. We have:

$$\vec{N} = N\hat{\jmath}\,, \qquad \vec{f} = -f\hat{\imath}\,, \qquad \vec{F}_g = mg\sin\theta\hat{\imath} - mg\cos\theta\hat{\jmath}\,. \tag{4.12}$$

Note also that we are assuming that the block is at rest and so its acceleration is $\vec{a} = 0$, and net force is also 0. Therefore, the net force in each dimension must be 0. As a vector equation, we have

$$\vec{N} + \vec{f} + \vec{F}_g = 0 = (mg\sin\theta - \mu_s N)\hat{\imath} + (N - mg\cos\theta)\hat{\jmath}\,, \tag{4.13}$$

or that

$$N - mg\cos\theta = 0\,, \qquad mg\sin\theta - \mu_s N = 0\,. \tag{4.14}$$

This requires that $N = mg\cos\theta$, and plugging this into the second equation, we have

$$mg\sin\theta = \mu_s mg\cos\theta\,, \qquad \text{or that} \qquad \mu_s = \tan\theta_{\rm min}\,, \tag{4.15}$$

where $\theta_{\rm min}$ is the minimum angle at which the block slides.

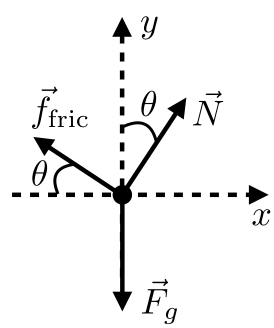
Whenever you get a result, you should always ask yourself if it makes sense. Do you ski, mountain climb, or hike? The tools you use for each of these activities are optimized to increase or decrease friction. First, if you ski, you wax your skis to decrease the coefficient of friction so you can ski faster, as well as ski (that is, actually move) on slopes with small inclines. If $\theta_{\rm min}$ is small, $\tan\theta_{\rm min}$ is also small and μ_s is small. Makes sense.

By contrast, if you are rock climbing, you don't want shoes covered in Teflon, you want sticky, rubber shoes. You want to stick to steep walls, making $\theta_{\rm min}$ as large as possible. If $\theta_{\rm min}$ is large, then μ_s is large. Again, makes sense.

4.2.3 Coefficient of Static Friction, Redux

For the final part of this analysis, I want to revisit the system we just studied, to explicitly demonstrate that indeed the physics is independent of our description of it. Instead of aligning our axes with the slope, we'll just keep the axes horizontal and vertical. This will lead to different intermediate steps, but the final result will be identical. Further, it is important to know how to solve physics problems in many different ways. Different solutions to a problem provide different insights into how the physics is working and manifesting itself, and can lead to a deeper understanding, even for the most mundane of problems.

So, with that motivation, let's redraw our free-body diagram with our new coordinate basis:



In this coordinate basis, the components of the vectors are

$$\vec{F}_g = -mg\hat{\jmath}\,, \qquad \vec{N} = N\sin\theta\hat{\imath} + N\cos\theta\hat{\jmath}\,, \qquad \vec{f} = -\mu_s N\cos\theta\hat{\imath} + \mu_s N\sin\theta\hat{\jmath}\,. \tag{4.16}$$

As before, there is no acceleration in either dimension so Newton's second law implies that both

$$N\sin\theta - \mu_s N\cos\theta = 0\,, \qquad -mg + N\cos\theta + \mu_s N\sin\theta = 0\,. \tag{4.17}$$

Now, in this coordinate system, the first equation, no acceleration in the horizontal dimension, immediately implies that

$$\mu_s = \tan\theta_{\text{min}}\,. \tag{4.18}$$

Recall that earlier we had found that $N = mg\cos\theta$, when our coordinates were oriented along the ramp. Does that still work in this case? Solving for N in the second equation, no acceleration in the vertical dimension, we have

$$N = \frac{mg}{\cos\theta + \mu_s \sin\theta} \,. \tag{4.19}$$

With $\mu_s = \tan\theta$, note that

$$\cos\theta + \tan\theta\sin\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta} \,. \tag{4.20}$$

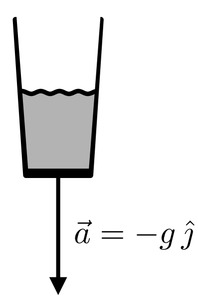
Then, $N = mg\cos\theta$, as earlier! So, indeed, the physics is independent of our description of it. A pithy way to state this is:

> Our art must imitate Nature, but Nature cannot imitate art.

4.3 Centripetal Force

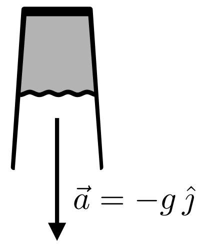
I want to impress my son, so I have thought of a fun experiment. We all know that water will sit in the bottom of an upright glass. Indeed, if this were not the case, life would be much more difficult. Anyway, that's not the experiment. What I want to do is keep the water in the glass when it is upside-down. Is this possible? Clearly, I can't just turn the water glass upside down and keep water in it, because that is how you drink. So, let's go systematically through systems that would keep the water in the glass.

First, let's pour water in the glass and just drop it, upright, on the ground. Does the water stay in the glass in this case? As always, we ignore air resistance. As the glass and water fall, they both accelerate at g:



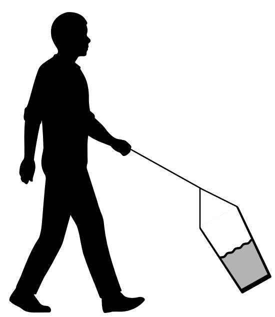
Because the glass and water start together and accelerate at the same rate, they stay together as they fall. Thus, the water stays in the glass as it falls.

This correspondingly tells us how we can keep water in the glass when it is upside-down. Pour water into a glass when you are on an airplane, jump out, and then turn the glass over! The water and glass (and you) will all accelerate at g, and so the water will stay in the glass:



So, we have succeeded! We just need to accelerate the glass/water downward at g, i.e., in free fall, and the water will stay in the glass. However, demanding that we do this by jumping out of a plane isn't very practical.

Let's think of another way to get the glass upside-down without the water spilling. We could also swing the glass with water in it in a big, overhand circle. At the bottom of the circle, the glass is upright with the water in it, while at the top of the circle, the glass will be upside-down with the water in it, just like we want. So, the physical set-up is



where I have (poorly) drawn a picture of me attempting to twirl the glass in a circle.

So, is it possible to twirl a glass in an overhand circle so that the water does not fall out?

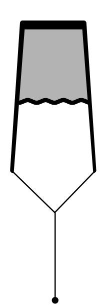
(a) Not possible

(b) Yes, possible at any rate

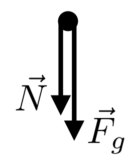
(c) Yes, possible but there is a minimum twirl rate

What do you think?

For the water to stay in the glass, then necessarily the water must stay in the glass when it is at the top of the circle:



What are the forces on the water? Of course gravity, but also (possibly) a normal force from the glass



There are no other relevant forces, so this suggests that the water is accelerating, by Newton's second law. We had already argued that for the water to stay in the glass, the acceleration a_y of the water must be (at least) g. Using Newton's second law we find

$$ma_y = N + mg\,, \qquad \text{or that} \qquad a_y = \frac{N}{m} + g \geq g\,, \tag{4.21}$$

as normal force and gravity act in the same direction.

Additionally, the glass/water system is traveling in a circle, so we know how to interpret this acceleration: it is centripetal acceleration, where its magnitude is

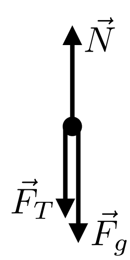
$$a_{\text{cent}} = a_y = \frac{v^2}{r}\,, \tag{4.22}$$

where v is the tangential velocity of the glass/water, and r is the length of the string I am swinging (the radius of the overhand circle). So, to keep the water in the glass, we must have that

$$a_y = \frac{v^2}{r} \geq g\,, \qquad \text{or that} \qquad v \geq \sqrt{rg}\,. \tag{4.23}$$

That is, there is a minimal speed below which the water will fall out of the glass and above which it will stay in.

Let's continue to analyze this system. In particular, what is the tension in the string at the top of the circle? To study this, let's consider the free-body diagram for the glass at the top:



There are only three forces acting on it at the top: its weight (of course), the tension in the string that twirls it, and the normal force of the water. The normal force of the glass on the water is a force pair with the normal force of the water on the glass, and so this force on the glass acts in the vertical direction, upward. By Newton's second law, the sum of these forces is responsible for the centripetal acceleration of the glass:

$$T + mg - N = ma_{\text{cent}} \geq mg\,. \tag{4.24}$$

In the inequality on the right, we have simply noted that the centripetal acceleration at the top of the loop must be at least the acceleration due to gravity, g. So, rearranging the inequality, we find that the magnitude of the tension force is at least the magnitude of the normal force:

$$T \geq N\,. \tag{4.25}$$

We argued earlier that the minimum normal force on the water/glass by the glass/water is $N = 0$. That is, the tension T is, well, anything and the glass can still travel in a circle.

However, our analysis assumed that there was a tension at all! What happens if it is no longer true that

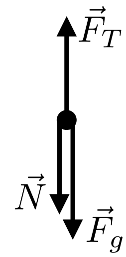
$$a_{\text{cent}} = \frac{v^2}{r} \geq g\,? \tag{4.26}$$

That is, what if the velocity of the glass is too small at the top of the loop? In that case, there is no normal force, the string goes slack, and the only force on the glass is gravity. That is, the glass would enter free fall, and just travel in a parabola, rather than a circle.

Let's test these predictions out! I have a cylinder on a tray and I will fill the cylinder with water and we'll observe what happens when I swing it above my head. These demonstrations are rather dangerous, especially with the potential for flying cylinders once they enter free fall. However, I am here to sacrifice my body to science, so let's do it! (See `https://youtu.be/hN5I1vqGaxU`)

4.3.1 Limits of Circular Motion and Centripetal Acceleration

It's also interesting to consider the forces that are acting on the water/glass at the bottom of the circle. At that point, the forces on the glass are



Now, if the glass is traveling in a circle, the sum of these forces are responsible for centripetal acceleration,

$$ma_{\text{cent}} = m\frac{v^2}{r} = T - N - mg\,, \tag{4.27}$$

or that the tension in the string is

$$T = m\frac{v^2}{r} + N + mg\,. \tag{4.28}$$

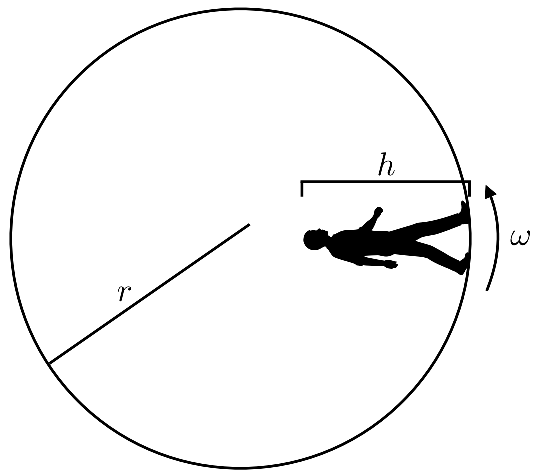
This is fascinating: we can spin/twirl the glass faster and faster (increase v) and eventually the tension will be so large that the string will break. Let's try this out! Actually, no way, this is much too dangerous.

Before moving on, I want to have you think about something. If the centripetal acceleration at the top of the loop of radius r is just g, then what could the expression for the velocity at the bottom of the loop be? What changes from when the glass is at the top to when it is at the bottom?

In a related vein, consider swinging on a, well, swing. Is it possible for you, with no one pushing you, to ever "swing over the bar"? Can you ever pump your legs hard enough to provide high enough speed such that your velocity at the tippy-top provides enough centripetal acceleration for equal g? How fast would you have to be going at the bottom?

A few more notes before we move on to the next chapter. One aspect of roller coasters is the feeling of weightlessness at the top of a loop. "Weightlessness" simply means that the only force acting on you is gravity, so you are accelerating at g. One feature of the particular feeling of weightlessness is "butterflies in your stomach." This is due to your organs literally floating in your body when you are weightless. When you are standing on ground, your organs are held in place by a matrix of ligaments and such, but when weightless, then tension on the matrix vanishes, leaving your guts just afloating.

Additionally, this property of circular motion can be exploited to simulate the force of gravity. There's a famous scene in *2001: A Space Odyssey* in which David Bowman is running on the inside of a revolving cylinder as such:



Say the revolving cylinder has radius r and is rotating with angular speed ω. We then know the centripetal acceleration of the cylinder:

$$a_{\text{cent}} = \omega^2 r\,. \tag{4.29}$$

If this equals g, then the acceleration of the cylinder would be the same as the acceleration due to gravity on Earth. However, would this apparatus actually simulate the gravity we know and love? Think about it!

Chapter 5

Energy

Beginning in this chapter, we will start our foray into **conservation laws**, their consequences, and utility for understanding physical systems. I had introduced conservation laws at the beginning of the lectures from asking why we can trust our memories. The physics answer to this is that the laws of physics, i.e., the rules that govern how we engage with Nature and our environment in particular, do not change in time. That is, things we learn yesterday (I hurt my hand by touching a hot stove) can be applied to actions tomorrow (don't touch a hot stove). I had also mentioned then that this implied the existence of a symmetry: a transformation that we can perform on a system that leaves it unchanged. More precisely, a symmetry transforms a system to itself. In the case of trusting our memories, the symmetry action is time translation: the laws of physics are unchanged by travel (translation) through time. Additionally, I had argued that there should be a single quantity that is a measure of this time translation symmetry. That is, if time translation is a perfect symmetry, then this quantity is conserved, its value does not change in time. This intricate relationship between symmetries and conservation laws is called Noether's theorem and the conserved quantity associated with time translation is **energy**.

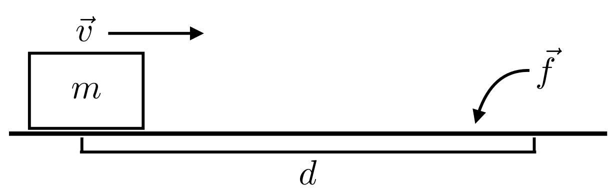
We denote energy by E and we will define it (for now) as the measure of an object's or system's ability to perform a task. This definition is consistent with our colloquial use of "energy." If you "have no energy," then you can't even perform simple tasks. Today, we will just study the energy of single object or particle systems, which will simplify our task for determining what this "energy" is.

5.1 Kinetic Energy

Historically, the first person to recognize conservation of energy as a consequence of Newton's laws was Émilie du Châtelet, an 18th century natural philosopher who also translated Newton's *Philosophiæ Naturalis Principia Mathematica* into French, making significant corrections and improvements on its presentation and implications. The method that du Châtelet, or other Enlightenment natural philosophers, used to measure the energy of a moving object was the following. A lead ball of mass m was thrown with speed v at a chunk of clay. The ball correspondingly smushed the clay, embedding itself a distance d into the clay. The distance d was thus a measure of the lead ball's ability to perform a task; that task being deforming the clay. We'll study a similar system, more familiar for what we have studied, to identify the energy of a moving object.

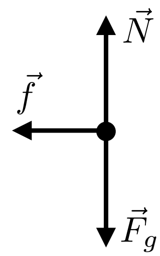
5.1.1 Energy from Newton's Laws

What we will do is the following. We will give a block of mass m a velocity $\vec{v}$, and then it will travel over a surface which has friction. This friction force will accelerate the block, eventually stopping it. We would like to determine the block's ability to slide over the surface. We will measure this ability, i.e., the block's energy, by the distance d it travels over the surface. So, the setup is



Before we study this system, I want to make a note. Energy is a **scalar** quantity: it has only a magnitude, and has no direction. From our setup, this makes sense: the distance d is just a distance (magnitude) and not a vector (displacement). Whatever direction $\vec{v}$ is, we want to stop it, so direction is irrelevant.

To study this system, let's recall the kinematic equations and Newton's second law. The free-body diagram for the block when it is on the surface with friction is



The block is not accelerating vertically, so $\vec{N} = -\vec{F}_g$. Newton's second law in the horizontal direction is

$$\vec{f} = -f\hat{\imath} = m\vec{a} = ma_x\hat{\imath}\,, \tag{5.1}$$

or that acceleration a_x is

$$a_x = -\frac{f}{m}\,. \tag{5.2}$$

We could relate f to normal force and therefore weight, but we won't here.

With the initial velocity vector $\vec{v} = v\hat{\imath}$, we have the kinematic distance equation

$$d = \frac{1}{2}a_x t^2 + vt\,, \tag{5.3}$$

as the block slides a distance d along the surface. This is written with time explicitly, but we can eliminate that using the kinematic equation for speed:

$$0 = a_x t + v\,, \tag{5.4}$$

where we note that the final velocity is 0. I want to emphasize that these kinematic equations can be used because acceleration a_x is constant: the friction force f is constant. Then, the time over which the block slides is

$$t = -\frac{v}{a_x}\,. \tag{5.5}$$

Plugging this into the equation for distance, we have

$$d = \frac{1}{2}a_x\left(-\frac{v}{a_x}\right)^2 - v\,\frac{v}{a_x} = -\frac{1}{2}\frac{v^2}{a_x}\,. \tag{5.6}$$

We had also found that $a_x = -f/m$ earlier, so this is also

$$-\frac{1}{2}\frac{v^2}{\left(-\frac{f}{m}\right)} = d\,, \tag{5.7}$$

or, more naturally, as

$$\boxed{\frac{1}{2}mv^2 = fd\,.} \tag{5.8}$$

This relationship is immensely profound. The expression on the left, $\frac{1}{2}mv^2$, is exclusively written in terms of the block's properties. It is called **kinetic energy** because it is a measure of the energy due to the block's motion:

$$K = \frac{1}{2}mv^2\,. \tag{5.9}$$

The expression on the right, fd, is exclusively written in terms of how the surface acts on the block. The surface exerts a force f on the block over a distance d. This force is responsible for reducing the kinetic energy of the block from $\frac{1}{2}mv^2$ to 0. As such, we say that the surface did **work** on the block equal to

$$W = fd\,. \tag{5.10}$$

We'll explore this later and more precisely, but work done by a force changes an object's kinetic energy. Specifically,

$$W = \Delta K\,, \tag{5.11}$$

called the **Work-Energy Theorem**.

5.1.2 Newton's Laws from Energy

Now, we said some words that if energy is conserved, then the laws of physics should be independent of time. The work-energy theorem is a statement of conservation of energy: kinetic energy can be transformed into some other form of energy by exerting work, but it can't disappear into the æther. Newton's second law is a law of physics, so if energy is conserved, it should somehow follow from $W = \Delta K$. Let's see how this is done in our example.

We had derived

$$\frac{1}{2}mv^2 - fd = 0\,, \tag{5.12}$$

which is just the statement of the work-energy theorem, with everything to one side. If this is true at any time, then it has no time dependence or its time derivative is also 0. That is, the statement of time independence is

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 - fd\right) = 0\,. \tag{5.13}$$

Let's take derivatives and see how we can simplify this expression. The mass m of the block is constant, so the only relevant derivative of kinetic energy is of the v^2 term. We find, using the chain rule,

$$\frac{dv^2}{dt} = \frac{dv^2}{dv}\frac{dv}{dt} = 2va\,, \tag{5.14}$$

where we note that

$$\frac{dv^2}{dv} = 2v\,, \qquad \text{and} \qquad \frac{dv}{dt} = a\,, \tag{5.15}$$

acceleration. Then, the time derivative of the kinetic energy is

$$\frac{d}{dt}\frac{1}{2}mv^2 = mva\,. \tag{5.16}$$

Next, the time derivative of fd is equally simple. By assumption, the friction force is a constant, so only d might depend on time. However,

$$\frac{d}{dt}\,d = -v\,, \tag{5.17}$$

the negative of the speed, because if d increases, speed decreases because friction slows the block. It then follows that

$$\frac{d}{dt}\,(fd) = -fv\,. \tag{5.18}$$

Using these results, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 - fd\right) = 0 = mva + fv\,. \tag{5.19}$$

The velocity v appears in both terms, so we can safely cancel it out, producing

$$ma = -f\,, \tag{5.20}$$

which is just Newton's second law for a friction force. This demonstrates that, in a well-defined way, Newton's second law is very literally derivative of conservation of energy. As such, we consider conservation of energy more fundamental than Newton's second law.

We will provide a more precise definition of work, the work-energy theorem, and demonstrate how Newton's second law follows as a vector equation from conservation of energy in later lectures.

5.1.3 Energies at Particle Collision Experiments

I now want to use this new idea of kinetic energy to understand a feature of my research. The Large Hadron Collider (LHC) in Geneva, Switzerland, accelerates and collides protons at enormous (relative) energies. We'll attempt to get a sense for how large the energy of an individual proton is at the LHC. First, the unit of energy in SI is called the **Joule** J, after James Joule, a Scottish engineer. By the work-energy theorem, note that the units of the Joule are

$$[\text{J}] = [\text{N}]\,\text{m} \;=\; \text{kg m}^2\text{ s}^{-2}\,. \tag{5.21}$$

In Joules, the kinetic energy of a proton at the LHC is about 10^{-6} J. (More useful units of energy in particle physics is the **electron-Volt**, for which the protons carry about 10^{13} eV of energy.) Objectively, is this a lot of energy?

10^{-6} J is a small number, but let's relate it to more everyday energies. For example, let's consider the kinetic energy of a flying mosquito. The mass of a mosquito is about 5 mg or 5×10^{-6} kg. At top speed, a mosquito can fly at about 1/2 m/s, so their kinetic energy is

$$K = \frac{1}{2}\left(5\times 10^{-6}\right)(0.5)^2\;\text{J} \approx 6\times 10^{-7}\;\text{J}\,, \tag{5.22}$$

very close to the kinetic energy of a single proton at the LHC! I want to emphasize the scale here. A mosquito contains about 10^{20} protons, and yet the LHC packs the same energy of a mosquito into 1 proton!

The energy of a mosquito may still be a bit abstract, so let's try another comparison. Your hand (rather, one of them) is about 0.5% of your body weight. I weigh about 80 kg (times g), so the mass of my hand is about

$$m_H = 0.5 \times 10^{-2} \times 80 \approx 0.4 \text{ kg}\,. \tag{5.23}$$

If the kinetic energy of your hand is K, then its velocity v is

$$K = \frac{1}{2} m_H v^2 \qquad \text{or, solving for } v, \qquad v = \sqrt{\frac{2K}{m_H}}\,. \tag{5.24}$$

Plugging in $K = 10^{-6}$ J and $m_H = 0.4$ kg, the velocity of one of your hands necessary to have kinetic energy equal to one proton at the LHC is

$$v = \sqrt{\frac{2 \times 10^{-6}}{0.4}} \text{ m/s} \approx 2 \times 10^{-3} \text{ m/s}\,, \tag{5.25}$$

or a couple of millimeters per second. This is about the rate of a (very) slow clap, again, contained in a single proton at the LHC.

The LHC doesn't collide individual protons together; rather, bunches of about 10^{11} protons are collided. There are about 10^{10} people on Earth, so there is more energy in the protons at the LHC than if energy human on Earth simultaneously clapped!

5.2 The Work-Energy Theorem

The work-energy theorem is the statement that the change in kinetic energy of an object is the amount of work done on that object:

$$\Delta K = W\,. \tag{5.26}$$

We had introduced kinetic energy as

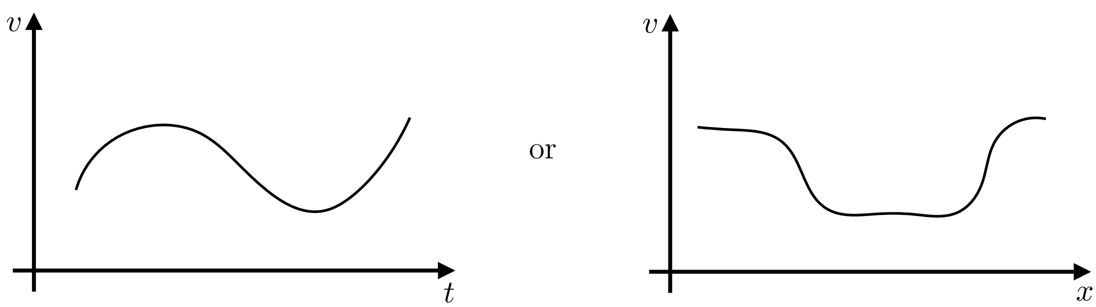
$$K = \frac{1}{2} m v^2\,, \tag{5.27}$$

and so

$$\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\,, \tag{5.28}$$

for initial and final velocities v_i and v_f, respectively. We had briefly introduced work, but we will make it more precise in this lecture.

Let's first consider motion in one dimension. Let's say that we have an object of mass m which is being acted on by a force F. This mass therefore has a velocity that varies as a function of time or position:



like so. We would like to derive a relationship of the change in kinetic energy ΔK of the object as it travels to the force acting on it.

The difference in kinetic energy at times t and $t + \Delta t$ is

$$\Delta K = \frac{1}{2}mv(t+\Delta t)^2 - \frac{1}{2}mv(t)^2\,. \tag{5.29}$$

As $\Delta t \to 0$, this can be related to the time derivative of kinetic energy, where

$$\frac{dK}{dt} = \lim_{\Delta t \to 0} \frac{\frac{1}{2}mv(t+\Delta t)^2 - \frac{1}{2}mv(t)^2}{\Delta t} = mva\,, \tag{5.30}$$

where $a = \frac{dv}{dt}$, and we used the chain rule. Note that $v(t)$ is velocity at time t, not velocity multiplied by t. Now, using Newton's second law, note that

$$mva = vF = F\frac{dx}{dt}\,, \tag{5.31}$$

where on the right, we just note that velocity is the time derivative of position x. Thus simply differentiating kinetic energy by time, we have found

$$\frac{dK}{dt} = F\frac{dx}{dt}\,. \tag{5.32}$$

Now for some trickery... In this Leibniz notation, derivatives are, really I swear, ratios, so we can cross cancel and divide. So, we "cancel" the dt factors, and find

$$dK = F\,dx\,, \tag{5.33}$$

or, by dividing by dx on both sides,

$$\frac{dK}{dx} = F\,. \tag{5.34}$$

That is, the derivative of kinetic energy with respect to position x is force!

Almost there; let's integrate both sides of this expression over position x from $x = a$ to $x = b$. The Fundamental Theorem of Calculus states that

$$\int_a^b \frac{dK}{dx}\,dx = K(x=b) - K(x=a) \equiv \Delta K\,, \tag{5.35}$$

while integrating over force, we find

$$\Delta K = \int_a^b F\,dx\,. \tag{5.36}$$

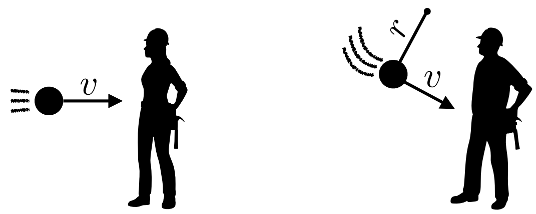
This is the work-energy theorem: the change in kinetic energy of an object is equal to the integral of force over the trajectory of the object. This implies Newton's second law and vice-versa.

5.2.1 Going Beyond One Dimension

That's the story in one dimension; how do we generalize the work-energy theorem to multiple dimensional motion? Let's think again about what kinetic energy is and how forces change it.

Example

Now, I want to introduce a definition of kinetic energy that will serve our purposes for considering the generalization of the work-energy theorem. I will define kinetic energy of a ball, for example, as measured by how much it hurts when it hits you. Let's consider two set-ups: one where the ball is thrown straight at you with speed v, and the other where the ball is attached to a string and rotated such that the ball has tangential speed v. That is, we have the set-ups:



Which ball will hurt more when it hits you?

(a) linear ball (b) circular ball (c) same hurt

Okay, now can I get a couple of volunteers? (Just kidding!)

Immediately before the ball hits you, they were both traveling with speed v. Who cares that one ball was traveling in a line and the other in a circle: all of that velocity, for both balls, has to stop when they hit you. So they will both hurt the same!

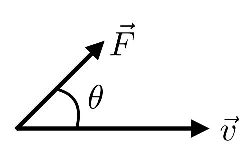
Why is this relevant? Well, are the accelerations of the balls different? What would that suggest about the forces acting on them? Correspondingly, what would that naively suggest for the work-energy theorem in multiple dimensions?

For the ball moving in a line, its kinetic energy is constant, $\frac{1}{2}mv^2$, until it hits you. There are no forces on the ball: its velocity vector is constant. For the ball moving in a circle, its velocity vector $\vec{v}$ is continuously changing direction, but keeps its speed constant. Apparently simply changing direction does not change kinetic energy. That is, accelerations and therefore forces that exclusively change the direction of velocity do no work.

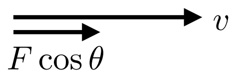
What special about the acceleration that keep the ball moving in a circle? It is centripetal acceleration, and as we discussed earlier is perpendicular to tangential velocity. That is, the force responsible for keeping the ball in a circle is perpendicular to the motion of the ball. This identification suggests that forces exerted perpendicular to motion only change direction, and do no work.

Therefore, to determine the work done on an object, we only care about those forces with components in the direction of motion. Only they can do non-zero work.

So let's analyze this for a particular velocity vector and force. Let's say we exert a force $\vec{F}$ on an object of mass m with velocity $\vec{v}$ as



Let's call the angle between the force and velocity θ. As argued earlier, the component of $\vec{F}$ perpendicular to $\vec{v}$ does no work, so just for studying energy, this is equivalent to



or that

$$\frac{dK}{dx} = F\,\cos\theta\,. \tag{5.37}$$

If we turn infinitesimals into small, finite changes, we have

$$\Delta K = F\,\Delta x\,\cos\theta = Fv\cos\theta\,\Delta t\,. \tag{5.38}$$

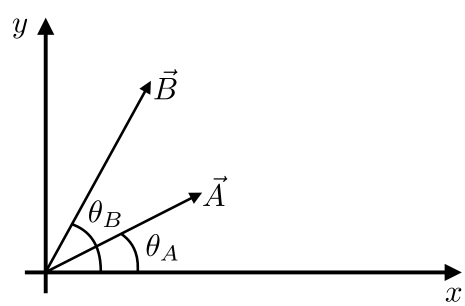
Note that if the object has speed v, then it travels a distance $\Delta x = v\,\Delta t$ in time Δt. The quantity $Fv\cos\theta$ picks out only the component of force in the direction of motion.

5.2.2 The Dot Product

It turns out that "$Fv\cos\theta$" can be nicely encoded in a vector operation called the **dot product**. Consider two, two-dimensional vectors $\vec{A}$ and $\vec{B}$. Without loss of generality, we can express their components as

$$\vec{A} = A\cos\theta_A\,\hat{\imath} + A\sin\theta_A\,\hat{\jmath}\,, \qquad \vec{B} = B\cos\theta_B\,\hat{\imath} + B\sin\theta_B\,\hat{\jmath}\,. \tag{5.39}$$

We have the picture that



Note that the angle between the two vectors is $\Delta\theta \equiv \theta_B - \theta_A$.

Now, the dot product is defined as the sum of the products of each components of the vectors. Specifically, we have

$$\vec{A}\cdot\vec{B} = (A\cos\theta_A)(B\cos\theta_B) + (A\sin\theta_A)(B\sin\theta_B) = AB(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B)$$

$$= AB\cos(\theta_B - \theta_A)\,. \tag{5.40}$$

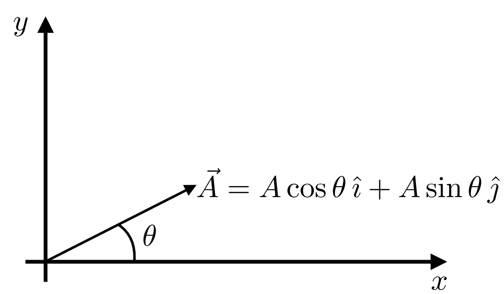
In the first line, we just multiplied x-components by x-components and summed them with y- times y-components. In the final line, we used a trigonometric identity. That is, the dot product is exactly what we need and picks out the shared component of two vectors.

So, we can equivalently write

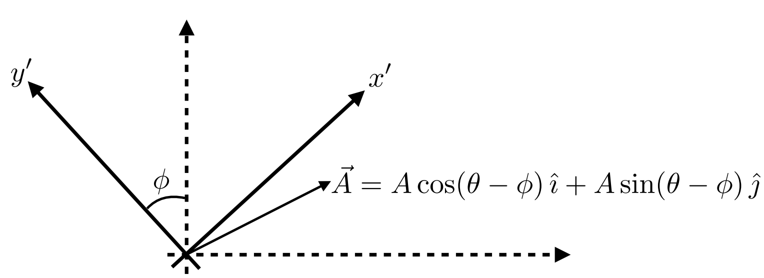
$$\frac{dK}{dt} = Fv\cos\theta = \vec{F}\cdot\vec{v}\,. \tag{5.41}$$

This is the work-energy theorem for motion in arbitrary dimensions.

There are a couple of other properties of the dot product that are useful to mention. First, a vector as a mathematical object is defined by how it changes when undergoing rotation. For instance, $\vec{A}$ vector with some coordinate system is

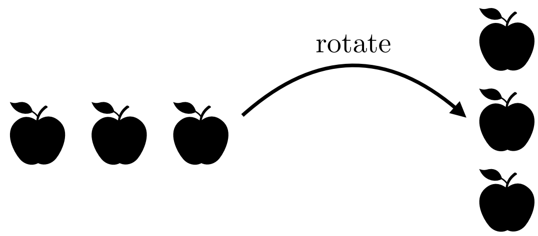


For coordinates rotated by ϕ with respect to these coordinates, the vector is then expressed as



Rotation changes the direction of the vector with respect to coordinate axes, but does not change its magnitude.

A scalar as a mathematical object is unchanged under rotation. Three apples are still three apples when rotated:



Let's consider the rotation of the dot product. For our vectors $\vec{A}$ and $\vec{B}$ from earlier, their angles with respect to the x-axis would transform under a rotation of ϕ as:

$$\theta_A \to \theta_A - \phi\,, \qquad \theta_B \to \theta_B - \phi\,. \tag{5.42}$$

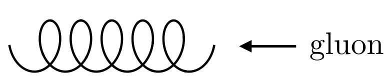
However, their dot product transforms as

$$\begin{aligned}\vec{A}\cdot\vec{B} &= AB\cos(\theta_B - \theta_A) \to AB\cos\left[(\theta_B - \phi) - (\theta_A - \phi)\right] \\ &= AB\cos(\theta_B - \theta_A) = \vec{A}\cdot\vec{B}\,.\end{aligned} \tag{5.43}$$

That is, the dot product is rotation-invariant. The dot product takes two vectors and returns a scalar, just a number.

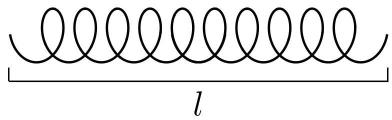
5.3 The Simple Harmonic Oscillator

In this lecture, we are going to introduce the approximation to end all approximations: the spring or **simple harmonic oscillator**. To first approximation, almost everything in Natural is modeled as a spring in physics. In my own research, interactions of elementary particles are modeled as mediated by a spring. In fact, the calculational technique used generally in particle physics called Feynman diagrams, after Richard Feynman who introduced them, uses a spring drawing to denote a gluon, the force carrier of the strong nuclear force:

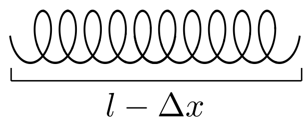


5.3.1 Model as a Spring

So what's the deal with springs and why are they everywhere? To answer this question, we need to figure out what force a spring can exert on an object. First, if you just encountered a spring on the street, it would likely be smoking Pall-Malls, wearing a trench coat; that is to say, it is relaxed. A relaxed spring is one for which it is neither extended nor compressed; it assumes a length l when no forces act on it

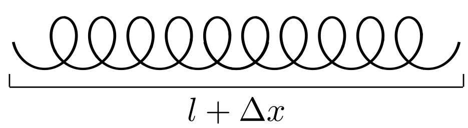


Now, let's imagine compressing the spring, from relaxed length l to $l - \Delta x$:

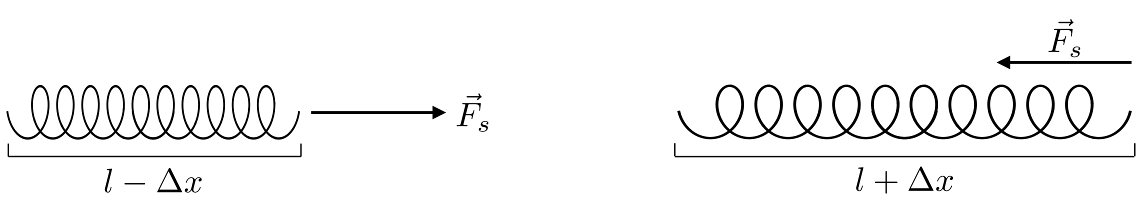


If you have ever tried to compress a spring, what do you notice about the force you must apply as Δx increases? It gets harder and harder to compress the spring as Δx increases! This is unlike gravity or friction forces that we have dealt with thus far: both of those (so far) have been constant forces, independent of position. Additionally, a compressed spring wants to return to the relaxed position. That is, if you compress a spring you feel a strong force pushing your hands apart.

One can do a similar thing with a spring extended to length $l + \Delta x$:



Now, to extend the spring, you have to pull your hands apart with more and more force as Δx increases. These observations suggest that the force a spring can exert on an object is monotonic with Δx. Further, the spring force is a **restoring force**: the force a spring exerts acts to return ("restore") a spring to its relaxed length l. That is, the force that the spring exerts is in the opposite direction of the change in length:



5.3.2 Hooke's Law

These considerations motivate **Hooke's law** for the force of an (ideal) spring:

$$\vec{F}_s = -k\,\Delta x\,\hat{\imath}\,, \tag{5.44}$$

for a spring oriented along the x-axis. Here, Δx is the difference in length of the spring currently and its relaxed length. k is called the **spring constant** and is a measure of the "stiffness" of the spring: larger k means stiffer spring (more force for the same Δx). The overall negative sign indicates that this is indeed a restoring force: the direction of force opposes the direction of compression ($\Delta x < 0$) or extension ($\Delta x > 0$).

Hooke's law is named after Robert Hooke, a contemporary of Newton. Hooke often used ciphers to disguise this scientific discoveries. He first described the law that now bears his name in a Latin anagram *ceiiinosssttuv*, whose solution is *Ut tensio, sic vis*, which translates to *As the extension, so the force*. Hooke also had a famous scientific rivalry with Newton, after criticizing Newton's theory of optics. As the president of the Royal Society, Newton saw to it that Hooke's scientific writings and even portraits of him were destroyed.

So why is the spring so universal in physics? It is because of its simplicity. A general function $f(x)$ can, under reasonable assumptions, be expressed in a polynomial-like form called a **Taylor series**,

$$f(x) = f(0) + x\,\frac{df}{dx}\bigg|_{x=0} + \frac{x^2}{2}\,\frac{d^2f}{dx^2}\bigg|_{x=0} + \cdots\,. \tag{5.45}$$

If x is small, near 0, then higher powers of x are small and can often be neglected. If x is small enough that we can neglect x^2 and higher terms and $f(0) = 0$, the function approximates to a line

$$f(x) \approx x\,\frac{df}{dx}\bigg|_{x=0}\,. \tag{5.46}$$

Hooke's law is simply the Taylor expansion of a force that depends on the relative displacement Δx, expanded to linear order in Δx:

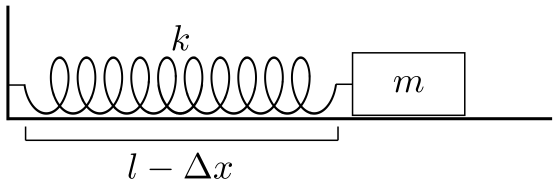
$$F(\Delta x) \approx \Delta x\,\frac{dF}{d(\Delta x)}\bigg|_{\Delta x=0}\,. \tag{5.47}$$

Hooke's law says that

$$\frac{dF}{d(\Delta x)}\bigg|_{\Delta x=0} = -k\,, \tag{5.48}$$

the spring constant. Springs, or Hooke's law, is so ubiquitous in physics because to first approximation (as defined by the Taylor series) almost every force anywhere is a spring force.

We recently learned about work, so let's see if we can figure out how much work a spring would do on an object. Let's say we have compressed a spring a distance Δx and at the end of the compressed spring we place a block of mass m. The block is on a frictionless surface and the other end of the spring is connected to a wall, to prevent it from flying away:



The spring constant is k. If we then let go of the block, how much work will the spring do on the block? First the force that the spring exerts on the mass is

$$\vec{F}_s = k\Delta x\,\hat{\imath}\,, \tag{5.49}$$

where we assume k and Δx are both positive. As the spring expands, pushing the block, it only acts to push the block up to the point it reaches its relaxed length, and then no longer pushes the block. Thus, the spring only exerts a force over the compressed distance Δx. The spring force is exerted in the direction of the block's motion, so the work done is

$$W = \int_0^{\Delta x} F_s\, d\Delta x' = \int_0^{\Delta x} k\,\Delta x'\, d\Delta x' = \frac{1}{2}k(\Delta x')^2\Big|_0^{\Delta x} = \frac{1}{2}k\,\Delta x^2\,. \tag{5.50}$$

This is an expression we will come back to over and over: the energy that a spring imparts on an object compressed (or expanded) by a distance Δx is $\frac{1}{2}k\,\Delta x^2$.

By the work-energy theorem, this is equal to the kinetic energy that the block gains

$$\Delta K = W = \frac{1}{2}k\,\Delta x^2 = \frac{1}{2}mv^2\,. \tag{5.51}$$

So the speed of the block after this springing is

$$v = \Delta x\sqrt{\frac{k}{m}}\,. \tag{5.52}$$

It is also useful to see how this follows from simple dimensional analysis. The units of the spring constant are

$$[k] = \left[\frac{F}{x}\right] = \left[\frac{ma}{x}\right] = MT^{-2}\,. \tag{5.53}$$

For a spring with constant k, Δx compressed distance, and pushing a mass m, the mass's speed after the spring push can be written as

$$v = k^a \Delta x^b m^c \,, \tag{5.54}$$

for three powers a, b, c. Expanding this expression out in basic units, note that

$$[k^a \Delta x^b m^c] = M^a T^{-2a} L^b M^c = M^{a+c} L^b T^{-2a} \,. \tag{5.55}$$

For this to equal speed v, which has dimensions

$$[v] = LT^{-1} \,, \tag{5.56}$$

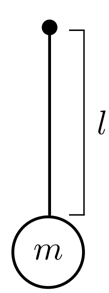
we must have that $a = -c$, $b = 1$, $a = \frac{1}{2}$, so then $c = -\frac{1}{2}$. That is, we find that

$$v = k^{1/2} \Delta x \, m^{-1/2} \,, \tag{5.57}$$

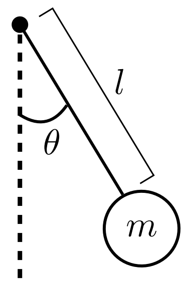
exactly as predicted by the work-energy theorem.

5.3.3 Model as a Pendulum

Springs aren't the only systems that exhibit Hooke's law. We will close this section by introducing the **pendulum**, the system of a suspended, swinging mass. Say a mass m is tied to the end of a string of length l like so

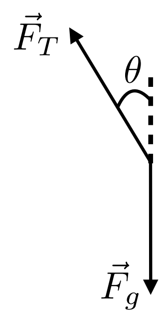


Now, we pull back the mass an angle θ from the vertical like

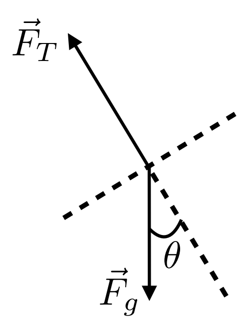


What is the net force on the mass right after we let go?

As always, let's draw our free-body diagram. The only forces acting on the mass are gravity and the tension in the string:



As the mass swings back and forth, it travels in a fixed radius trajectory, along the arc of a circle. Therefore, there must be some centripetal acceleration keeping the mass in this arc. This suggests that to analyze the forces, we should align an axis with the string and the other perpendicular to it. So we have



The net force in the direction of the string would be

$$F_T - mg\cos\theta = m\frac{v^2}{l}\,, \tag{5.58}$$

as this force is centripetal, and responsible for movement along a circular arc. The force perpendicular to this, tangent to the arc, is

$$-mg\sin\theta = ma_T\,, \tag{5.59}$$

where a_T is the tangential acceleration. This is equivalently expressed as

$$-mg\sin\theta = ma_T = m\frac{dv}{dt} = ml\frac{d\omega}{dt}\,, \tag{5.60}$$

where we note that $v = l\omega$, and ω is the angular velocity,

$$\omega = \frac{d\theta}{dt}\,. \tag{5.61}$$

That is, Newton's law tangent to the arc reduces to

$$-g\sin\theta = l\frac{d\omega}{dt} = l\frac{d}{dt}\frac{d\theta}{dt} = l\frac{d^2\theta}{dt^2}\,, \tag{5.62}$$

or that

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta\,. \tag{5.63}$$

Written as it is now, this is a bit hard to parse as that $\sin\theta$ factor is scary. However, if θ is a small angle, we can Taylor expand $\sin\theta$ as

$$\sin\theta = \sin 0 + \theta\left.\frac{d\sin\theta}{d\theta}\right|_{\theta=0} + \cdots = \theta + \cdots\,. \tag{5.64}$$

$\sin 0 = 0$, the derivative of $\sin\theta$ is $\cos\theta$, and $\cos 0 = 1$, and so, to lowest order in the Taylor expansion, $\sin\theta \approx \theta$. Using this, we find that

$$\boxed{\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta\,.} \tag{5.65}$$

The angular acceleration, denoted as α, of the pendulum,

$$\alpha \equiv \frac{d^2\theta}{dt^2}\,, \tag{5.66}$$

is linearly proportional to θ, and further is a restoring force, because of the negative sign.

Compare this to Newton's second law for a spring, where

$$-k\,\Delta x = ma = m\frac{d^2\Delta x}{dt} \qquad \text{or that} \qquad \frac{d^2\Delta x}{dt^2} = -\frac{k}{m}\Delta x\,. \tag{5.67}$$

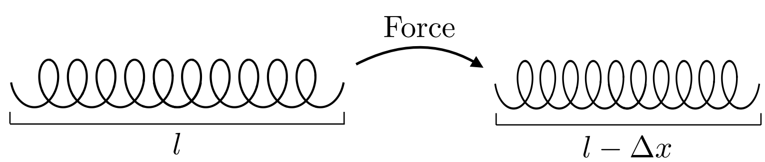
Apparently, then, a pendulum is just a spring. Let's see how this works! (See `https://youtu.be/2bSgoOOukWY`)

5.4 Potential Energy

Over the past several lectures, we have introduced the concept of energy, how it can be conserved, the work-energy theorem, and the work of a spring. In this lecture, we will discuss the important concept of **potential energy**, energy that is stored in an object or system that can be utilized later to perform some task (remember our definition of energy?).

5.4.1 Motivation From a Spring

A canonical example of potential energy is that stored in a compressed spring. Think about it: initially the spring is relaxed. To compress the spring and amount Δx, you have to exert a force on the spring over a distance Δx. Therefore, you did work on the spring.



However, the work you did to compress the spring didn't change the spring's kinetic energy as the spring is still at rest. However, you clearly exerted energy (that is, did work), that that energy can't have vanished into the æther. So, where did it go? The work you performed on the spring transferred to the spring's potential to do work on another object. As discussed last lecture, when compressed by an amount Δx, a spring will do work on a mass at the end of the spring of

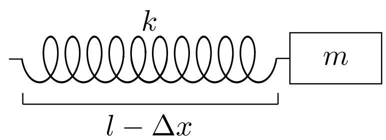
$$W = \frac{1}{2}k\,\Delta x^2\,, \tag{5.68}$$

where k is the spring constant.

The work you did on the spring to compress it is then stored in the potential ability for the spring to do an amount of work equal to $\frac{1}{2}k\Delta x^2$ on an object. More compactly, we say that a spring compressed by an amount Δx has potential energy U equal to

$$U = \frac{1}{2}k\Delta x^2\,. \tag{5.69}$$

If we put a box of mass m at the end of a spring which had potential energy of $\frac{1}{2}k\Delta x^2$, then this would be transferred into kinetic energy of the box once the spring returned to its relaxed length. If we thought of our universe as solely consisting of the block/spring system with no friction whatsoever, then the only types of energy allowed are the potential energy of the spring and the kinetic energy of the block. That is, for the system



if it is isolated or **closed**, then its total energy must be conserved as there is no way for energy to be lost or gained. Therefore, the sum of the kinetic and potential energy of this system is constant in time:

$$U + K = \frac{1}{2}k\Delta x^2 + \frac{1}{2}mv^2 = \text{ constant } \equiv E\,, \tag{5.70}$$

where E is the total energy. Demanding that the total energy be constant in time is typically vastly simpler for analyzing a problem than using Newton's second law, even though they lead to equivalent results.

5.4.2 Conservative Forces

Another thing to note about this spring potential energy is its simple relationship to Hooke's law. Let's take a derivative of U with respect to Δx:

$$\frac{dU}{d(\Delta x)} = \frac{1}{2}k\frac{d\Delta x^2}{d(\Delta x)} = k\Delta x = -F_{\text{spring}}\,, \tag{5.71}$$

by Hooke's law. That is, we note that

$$F_{\text{spring}} = -k\Delta x = -\frac{dU}{d\Delta x}\,. \tag{5.72}$$

Forces for which they are related to a potential energy by this negative derivative are called **conservative forces**. The name doesn't connote US political parties, but rather that the work done by such a force is exclusively from potential energy decrease (hence the "−" sign). Conservative forces, well, conserve energy all on their own.

We'll study another conservative force in a second, but it's important to note that not all forces are conservative. Perhaps the most familiar example is the force of friction. As we have discussed, friction can do work on an object to change its kinetic energy, but in doing so, that kinetic energy is transferred into many different forms of energy (heat, sound, etc.). The work friction does on an object does not exclusively turn that kinetic energy into potential energy, like with an ideal spring. We therefore say that friction is a **non-conservative force**. For friction, it is not possible to express it as a derivative of a potential energy.

Enough about non-conservative forces for now; let's get back to conservative forces and perhaps the most familiar force of all: gravity. First, let's argue that gravity is indeed conservative. Well, where does all of the work that gravity does on an object go to? Into changing the kinetic energy of the object! We can imagine a world without air, friction, etc., and gravity would still be doing its thing, keeping thrown balls in parabolic trajectories. Therefore, we can use the conservative force formula to determine the potential energy of gravity. Considering the force of gravity exclusively in one dimension (up and down), the force of gravity is

$$F_g = -mg\,, \tag{5.73}$$

when acting on an object of mass m. The corresponding potential energy is found from integrating from an initial height h_1 to a final height h_2:

$$U = -\int_{h_1}^{h_2} F_g\,dx = mg\int_{h_1}^{h_2} dx = mg(h_2 - h_1)\,. \tag{5.74}$$

That is, the gravitational potential energy is linear in the height difference from the current position to a reference position.

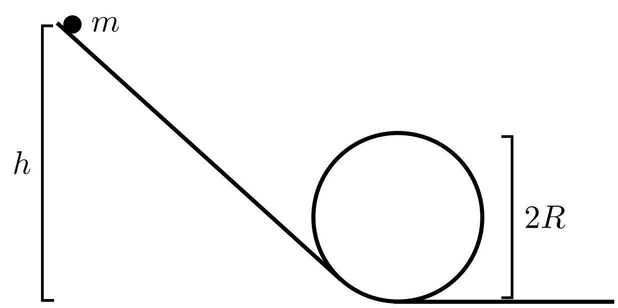
Unlike the potential energy of a spring, gravitational potential energy can be positive or negative in sign. This may seem weird, but all that matters are potential energy differences for determining how gravity affects an object's kinetic energy. That is, the statement of conservation of energy for an object of mass m exclusively acted on by gravity is

$$E = mgh + \frac{1}{2}mv^2 = \text{ constant}\,. \tag{5.75}$$

where h is the height above a vertical origin point.

5.4.3 Ball on a Loop-the-Loop

This is still a bit abstract, so let's consider a concrete, real system in which we can test this whole "conservation of energy" stuff. What I have in mind is a loop-the-loop set-up:



A ball travels down the ramp and then through the loop, ultimately traveling out, off to the right. We would like to predict the height h such that the ball stays on the track throughout traveling through the loop. The radius of the loop is R. We'll test this out once we have a prediction.

While we won't solve it this way, let's imagine that we attempt to solve with Newton's second law directly. Where do we even start? We would need free-body diagrams to determine the speed of the ball at the end of the ramp, which is easy enough. However, imagine the free-body diagrams for analyzing when the ball is in the loop. The direction and magnitude of the forces on the ball constantly change, so this would be a nightmare to analyze.

Luckily, we have energy on our side. We simply need to evaluate the initial energy and the final energy, equate them, and we can solve for height. So let's do this!

Initially, the ball is released from rest a height h above the ground. Initial kinetic energy is therefore 0, so the total initial energy is

$$E_i = mgh\,. \tag{5.76}$$

Now, if the ball is supposed to reach the top of the loop and remain on the track, two things must happen. First, the ball has to actually have enough energy to even reach a height of $2R$ (the top of the loop), but further, if the ball is still on the track, then it is traveling in a circle. As such, there must be a centripetal acceleration acting on the ball at the top of the loop. Gravitational force is always there, so, at least, the centripetal acceleration is g.

If the centripetal acceleration is g, then the ball has a minimal speed at the top of the loop:

$$a_{\text{cent}} = \frac{v^2}{R} = g\,, \qquad \text{or that} \qquad v_{\min}^2 = Rg\,. \tag{5.77}$$

This correspondingly implies that there is a minimal kinetic energy that the ball has at the top of the loop. This kinetic energy is

$$K_{\min} = \frac{1}{2}mv_{\min}^2 = \frac{1}{2}mRg\,. \tag{5.78}$$

The ball also has gravitational potential energy, as it has a non-zero height above the ground. This potential energy is

$$U = mg(2R)\,, \tag{5.79}$$

as the ball is a height $2R$ from the ground at the top of the loop.

So, the total energy of the ball at the top of the loop must be at least

$$E_{\text{top}} = K_{\min} + U = \frac{1}{2}mRg + mg(2R) = \frac{5}{2}mgR\,. \tag{5.80}$$

By conservation of energy, this has to equal the initial potential energy of the ball a height h above the ground. This therefore enables us to solve for this minimum height h as

$$E_i = mgh = E_{\text{top}} = \frac{5}{2}mgR\,, \tag{5.81}$$

or that $h = \frac{5}{2}R$. Let's try this out! Let's see if the ball indeed stays in the loop if the initial height is at least $\frac{5}{2}R$. (See `https://youtu.be/B5cSCnEyi28`)

5.5 Power

We have discussed energy, its conservation, kinetic versus potential, and work, and in this lecture we are going to tie together some loose ends before moving on. Earlier, we had discussed the notion of a conservative force, a force for which the work that it does exclusively comes from expending potential energy. The conservative forces we will focus on in this class are gravity and Hooke's law (springs), and we used this idea to determine the height to release the ball to go around a loop-the-loop. What if there are non-conservative forces in the game, like friction? Conservation of energy still holds, we just have to account for the work done by the non-conservative forces.

5.5.1 Conservation of Total Energy

Conservation of total energy is simply the statement that the energy measured at an initial time E_i is equal to the energy measured at some later time E_f:

$$E_i = E_f \, . \tag{5.82}$$

With only conservative forces in the ballgame, this can be restated through a sum of kinetic and potential energies:

$$K_i + U_i = K_f + U_f \qquad \text{(conservative forces only)}. \tag{5.83}$$

When analyzing a system in which the only forces are gravity and springs, this form of energy conservation is most useful.

With friction or other non-conservative forces around, some of that initial energy can be lost to heat, sound, etc., and not manifest as kinetic or potential energy in the final state. Therefore, accounting for this energy moved from a "useful" form (kinetic, potential) to a "useless" form (heat, sound), we account for the work that non-conservative forces did in going from the initial system to the final system:

$$K_i + U_i + W_{\text{non-cons}} = K_f + U_f \, . \tag{5.84}$$

Note that energy is still conserved, just not strictly conserved as kinetic and potential energy exclusively. Also most (all?) work by non-conservative forces is negative; e.g., friction slows an object. So this implies that, in general, when non-conservative forces are around, initial kinetic and potential energies are larger than their final values.

5.5.2 Differential Energy Delivered

Another key concept with energy is **power**, or energy used or delivered per unit time. At its simplest, power P is just the time derivative of the energy of some object:

$$P = \frac{dE}{dt} \, . \tag{5.85}$$

A single object like a car, horse, plane, etc., is not a closed system, so its energy does not need to be conserved; that is, it can change in time. In our everyday experience, it is power that makes a task challenging. Expending a lot of energy very quickly is more difficult than expending the same energy more slowly.

Let's derive another relationship of power using the work-energy theorem. The work-energy theorem states that

$$\Delta E = \int_a^b F\, dx\,, \tag{5.86}$$

that is, the energy of an object (spring, car, apple, etc.) changes if a force is applied over some distance $x \in [a, b]$. Differentially, this relationship is

$$dE = F\, dx\,. \tag{5.87}$$

Now, to relate this to power, we just divide by the infinitesimal time dt on both sides. (Nota Bene: I am not a mathematician, so questionable manipulations with infinitesimals are all cool!) That is,

$$\frac{dE}{dt} = P = F\,\frac{dx}{dt} = Fv\,, \tag{5.88}$$

or, that power is force times velocity. Now, I've been working in one dimension, so to generalize to multiple dimensions, we need a dot product between force and velocity:

$$P = \vec{F} \cdot \vec{v} = Fv\cos\theta\,. \tag{5.89}$$

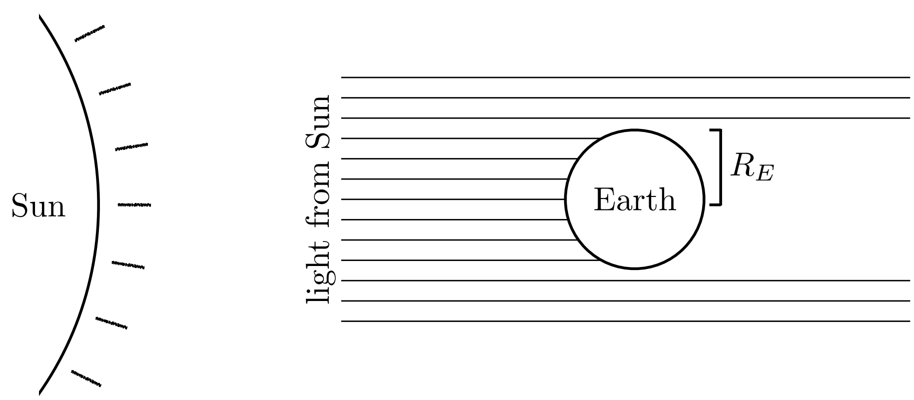
The units of power in SI are called *Watts*, after James Watt who invented the steam engine. A Watt is, not surprisingly, one Joule of energy per second, both SI units themselves. Another unit of power you might have heard of is *horsepower*, which interestingly was introduced by James Watt to compare the output of his steam engines to draft horses. The "horsepower" used in the United States for car energy output, for example, is 745.7 Watts.

5.5.3 Feasibility of Solar Power

Now, with this definition of power, I want to estimate the total power that is accessible from the sun on Earth. Solar power is increasingly becoming an important renewable resource, providing energy to us to charge our phones, run our trains, or heat our lecture halls. In this section, I want to estimate the possible solar power that we can harness on Earth.

First, I will tell you a couple of things. The power consumption of the entire Earth is about 10^{13} Watts, 10 TeraWatts. That means that every second, 10^{13} Joules are needed for everyone on Earth to heat their homes, run their Teslas, or cook their dinner. So, we're going to attemp to answer the question of whether solar power can account for these 10^{13} Watts.

The amount of power from light emitted by the Sun incident on Earth is about 1000 Watts per square meter. That is, when the Sun is directly overhead on a clear day, 1000 Joules of light energy hit a square meter of ground every second. For some context, a standard light bulb in your house might use about 60 Watts of power to run. (LED bulbs use much less power for the same light output, though.) So, as a first step in getting to our answer, let's consider how much solar power is incident on Earth at any given time. The trick to answer this is to introduce the notion of **cross-sectional area**. The sun shines light on Earth, and the cross-sectional area is the size of the shadow that the Earth casts:



The size of Earth's shadow is equal to the area of a circle with Earth's radius; this is called the cross-sectional area because if you cut Earth in half (i.e., made a *cross-section*), the area of the surface you opened up would be

$$\text{Area} = \pi R_E^2 \,. \tag{5.90}$$

With the radius of the Earth $R_E \approx 6000$ km $= 6 \times 10^6$ m, the cross-sectional area of Earth is

$$\text{Area} = \pi (6 \times 10^6)^2 \text{ m}^2 \approx 10^{14} \text{ m}^2 \,. \tag{5.91}$$

Again, as with everything in this game, orders of magnitude are sufficient. Using this result to find the total solar power incident on Earth, we multiply this area by the 1000 W/m^2 to find

$$P = 1000 \text{ W/m}^2 \cdot 10^{14} \text{ m}^2 \approx 10^{17} \text{ W} . \tag{5.92}$$

This would seem to be totally enough to power all of Earth, with orders of magnitude to spare. But, there's a catch. To capture all of this power, we would have to cover the entire Earth in solar panels, the sky would always have to be clear, and solar panels would have to be 100% efficient. But none of these are true, so we need to incorporate realistic numbers in an estimate.

First, the efficiency of commercial solar panels is about 10%. That is, for an incident power P on a solar panel, only about 1/10 of that power can actually be turned into electricity to power a toaster. So, out of the 10^{17} W of solar power incident on Earth, we can only extract about 10^{16} W for our use.

Now, we can't really hope to cover the ocean with solar panels. Oceans cover about 70% of the Earth's surface, or land is only about 30% of Earth's surface. Further, clouds cover about 70% of Earth's surface at any given time, so of the 30% that is land, only about 30% of it has a clear shot of the sun. 30% of 30% is about 10% again, so restricting solar panels to be on land means that there is only about 10^{15} W of power for our use.

Continuing, we can't actually cover all land with solar panels. If we did, no light would hit the ground, so there would be no farms, no forests, no fields. However, we could imagine that, say, a solar panel was installed on the roof of every building on Earth. This eliminates or at least minimizes the further footprint on the environment. As an estimate of the area of roofs on Earth, the total fraction of land area that is urban is about 3%. Of course, all of an urban area isn't just roofs, so perhaps a tenth of urban areas could be covered in solar panels. Including this factor of about 0.3% or 3/1000, the total area where solar panels could be reduces the total solar power accessible to use to about

$$P_{\text{solar}} \approx 3 \times 10^{12} \text{ W} . \tag{5.93}$$

There may be further constraints on the total solar power that can be harnessed, like infrastructure issues, but even now, we have fallen below the level of covering all power needs of Earth. Solar can't be all if we are to divest energy consumption from petroleum to renewable sources.

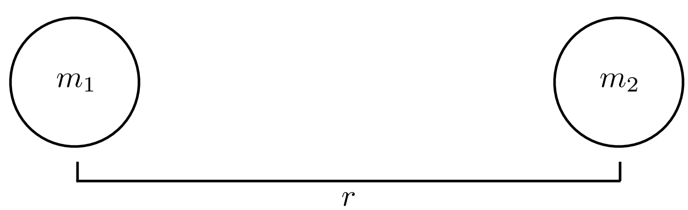
Chapter 6

Gravitation

In this chapter, we are going to create a theory of gravity that generalizes our simple discussion of a uniform force, universally pulling objects toward the ground. This theory that we will create will subsume our constant accelerating gravity, and explicitly predict when that assumption breaks down.

6.1 Derivation of Universal Gravitation

To construct this theory of gravity, we will imagine placing two objects of mass m_1 and m_2 a distance r apart:



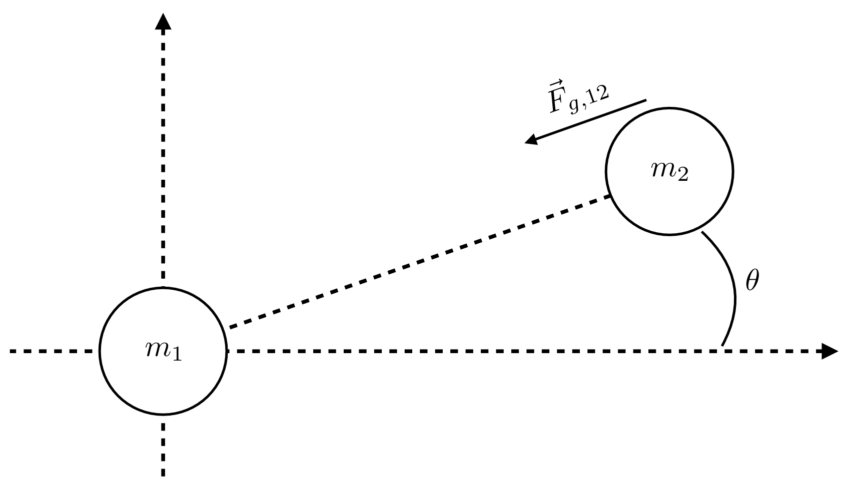
We would like to determine the force of gravity from mass 1 on mass 2, $\vec{F}_{g,12}$. Note also that we assume that these two masses are balls, but we will actually work in the approximation that they are points, and have no spatial extent. If point masses make you uncomfortable, then equivalently we work in the limit in which the distance between the objects r is much larger than either of their individual radii:

$$r \gg r_1, r_2\,. \tag{6.1}$$

6.1.1 Universal Attraction

The gravitational force on 2 by 1 is a vector, so we need to determine both its magnitude and direction. Let's start with the direction. Gravity is a universally attractive force, meaning that two masses are always attracted to one another via gravity. Specifically, in the case at hand, the direction of the force on mass m_2 points toward m_1:

Now, with this particular alignment, $\vec{F}_{g,12} = -F_{g,12}\hat{\imath}$, where $F_{g,12}$ is the magnitude of $\vec{F}_{g,12}$. However we can orient our axes to describe the positions of the two masses however we want. A convenient orientation is with m_1 at the origin and m_2 an angle θ above the horizontal, when projected on two dimensions:



With this orientation, the gravitational force is

$$\vec{F}_{g,12} = -F_{g,12} \left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right) . \tag{6.2}$$

Regardless of θ, the gravitational force vector points toward the origin, along a line that emanates from the origin. Such lines are nothing more than radial lines (they "radiate" from the origin) and we denote the vector with unit length that points along a radial line as $\hat{r}$:

$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} . \tag{6.3}$$

Another way to say this is that there is a rotational symmetry about mass m_1: rotating mass m_2 any angle about m_1 leaves the magnitude of gravitational force the same, and just rotates its direction to always point toward m_1. Thus, the gravitational force is

$$\vec{F}_{g,12} = -F_{g,12}\hat{r}\,. \tag{6.4}$$

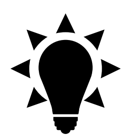
6.1.2 Inverse Square Law

Okay, we have the direction; what about magnitude? In general, as they are given quantities, the gravitational force could depend on distance r and the masses m_1, m_2 via

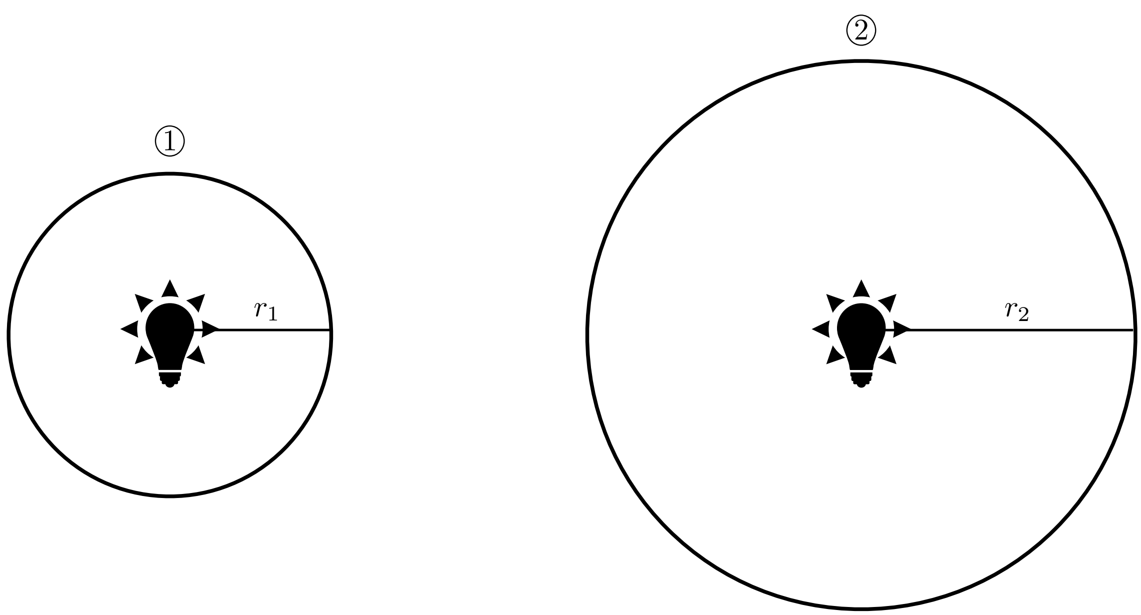
$$F_{g,12} \equiv F_{g,12}(r, m_1, m_2)\,. \tag{6.5}$$

Let's focus on the distance dependence first.

We live in three spatial dimensions, which may be obvious, but is extremely important for determining the dependence on r. We will work by analogy here, first imagining a light bulb hanging out in space:



This light bulb emits light in all directions uniformly. Let's imagine putting the bulb inside a sphere of radius r_1 and r_2 with $r_1 < r_2$:



How does the total amount of light that is captured by the two spheres compare?

(a) Sphere 1 more light (b) Sphere 2 more light (c) Same amount

The bulbs in both cases are identical, outputting the same amount of light and both spheres capture all of the light from the bulb. Therefore, they capture the same amount of light.

However, imagine that you are sitting on the interior surface of the spheres. In which case would the bulb appear brighter?

(a) Sphere 1 is brighter (b) Sphere 2 is brighter (c) Same brightness

Now, your eye is not like the sphere; it does not capture all of the light emitted by the bulb. Your eye only captures the light that hits a very small region. The amount of light that hits a given small region of the sphere is controlled by the total amount of light from the bulb divided by the surface area of the sphere, the light per unit area. The surface area of a sphere is

$$A = 4\pi r^2 \,, \tag{6.6}$$

where r is its radius. Thus, the light per unit area for the two sphere is

$$\frac{L}{A_1} = \frac{L}{4\pi r_1^2} \,, \qquad \frac{L}{A_2} = \frac{L}{4\pi r_2^2} \,, \tag{6.7}$$

where L is a measure of the total amount of light from the bulb. Because $r_2 > r_1$, the light per unit area for sphere 2 is smaller than for sphere 1, so you perceive the light in that case as dimmer. Note that the perceived brightness follows an **inverse square law**: if the radius of the sphere doubles, the perceived brightness decreases by a factor of four.

Now, let's take this observation to understand gravity. Our universe with mass 1 and mass 2 is still three-dimensional, and as we have emphasized throughout this class, forces always have some agent. Let's hypothesize that this agent for gravity is similar to the light from the bulb. That is, the total effect of gravity from mass m_1 is constant and only depends on properties of mass m_1. However, the density of the agent that exerts gravitational force on mass m_2, like the light, would decrease like $1/r^2$. Thus, with this hypothesis, the force of gravity would also follow an inverse square law:

$$F_{g,12}(r, m_1, m_2) = \frac{1}{r^2}\, M(m_1, m_2) \,, \tag{6.8}$$

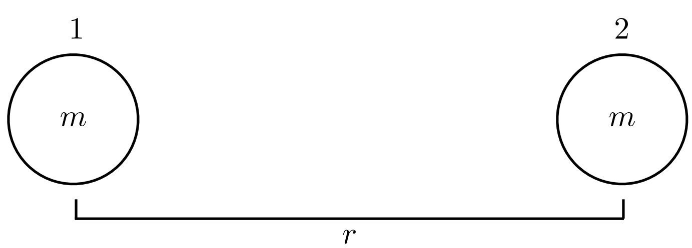
where $M(m_1, m_2)$ is a function purely of the masses m_1, m_2.

Again, in science we don't need to answer "why?" for every question to make progress. We can hypothesize and test our hypothesis and learn something about the universe. We don't need to answer the question of what the gravitational force agent is to test our theory of gravity.

6.1.3 Linearity in Masses

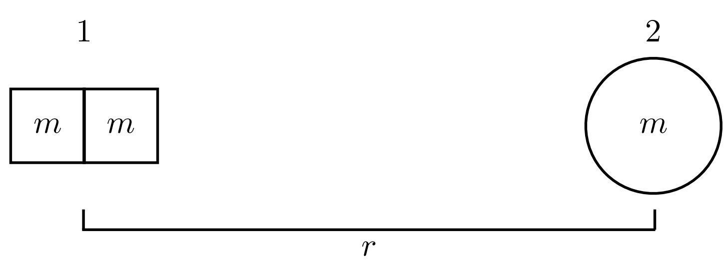
Now, let's figure out the mass dependence of the gravitational force. We use the equivalence principle, so gravitational mass and inertial mass are equivalent and just "mass." Mass is a measure of how much "stuff" an object is constructed from (I don't know what "stuff" is, however). So, we will answer the question of how the amount of stuff affects gravity.

To proceed, we will additionally assume that the effects of gravity are linear: that is, the net gravitational force on an object by two objects is simply the sum of individual forces. We will use this in a second. First, let's imagine that $m_2 = m_1 = m$, some basic unit of mass. Then, from what we have constructed above, the gravitational force in this case is



$$\Rightarrow F_{g,12} = \frac{M(m,m)}{r^2}$$

Now, let's imagine making $m_1 = 2m$, keeping $m_2 = m$. We can visualize this as

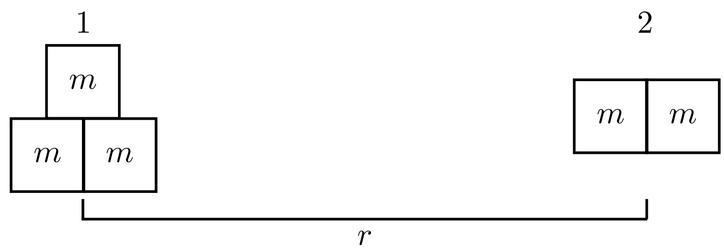


where we have imagined putting two blocks of mass m each at the location 1. By linearity of gravity, to find the force on mass 2, we can sum together the gravitational forces of the blocks individually:

$$F_{g,12} = \frac{M(m,m)}{r^2} + \frac{M(m,m)}{r^2} = \frac{2M(m,m)}{r^2}\,. \tag{6.9}$$

Note that we don't have to worry about vector addition because we assume that the blocks are at the same point.

Continuing, let's imagine that $m_1 = 3m$ and $m_2 = 2m$:



Using the linearity of gravity, what is the force on m_2?

(a) $F_{g,12} = \frac{3M(m,m)}{r^2}$ (b) $F_{g,12} = \frac{6M(m,m)}{r^2}$ (c) $F_{g,12} = \frac{2M(m,m)}{r^2}$

If $m_1 = 3m$ and $m_2 = 2m$, then there are a total of 6 pairs of masses in which one comes from m_1 and the other in the pair comes from m_2. 6 is simply the product of the relative masses of m_1 and m_2:

$$\frac{m_1}{m}\frac{m_2}{m} = \frac{3m}{m}\frac{2m}{m} = 6\,. \tag{6.10}$$

So, generalizing, if $m_1 = N_1 m$ and $m_2 = N_2 m$, where N_1 and N_2 are positive numbers, the gravitational force on mass 2 is

$$F_{g,12} = \frac{N_1 N_2 M(m,m)}{r^2}\,, \tag{6.11}$$

or we can express it as

$$F_{g,12} = \frac{G_N m_1 m_2}{r^2}\,, \tag{6.12}$$

proportional to the product of masses m_1 and m_2.

G_N is a constant of proportionality, called **Newton's constant**, to ensure that units are correct. With force having units of kg m/s^2, the units of G_N are

$$[G_N] = \left[\frac{Fr^2}{m^2}\right] = MLT^{-2}L^2M^{-2} = M^{-1}L^3T^{-2}\,, \tag{6.13}$$

or in SI, $[G_N] = \text{kg}^{-1}\text{m}^3\text{s}^{-2}$. Thus, the force vector between two masses is

$$\boxed{\vec{F}_{g,12} = -\frac{G_N m_1 m_2}{r^2}\,\hat{r}\,.} \tag{6.14}$$

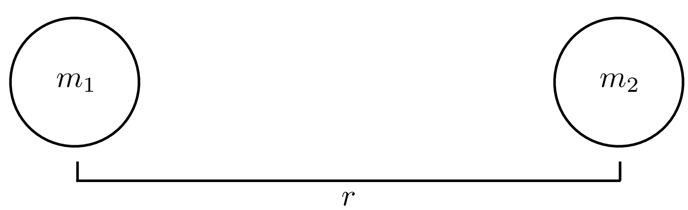
This is called Newton's **universal law of gravitation**. The value of Newton's constant is $G_N = 6.67 \times 10^{-11}\ \text{kg}^{-1}\text{m}^3\text{s}^{-2}$, and this sets the strength of the gravitational force between two masses. If this value were larger, the force would be larger, and if it were smaller, the force would be smaller. The fact that it is of order 10^{-11} in SI units means that the strength of gravity is *very* weak. The entire mass of the Earth pulls you down, but you can still jump up, off Earth, using your measly legs!

6.2 Gravitational Potential Energy

Let's now move on to studying gravitational force in another way, to expose the energy that it can store. This will also segue into one of the most mysterious objects in the universe.

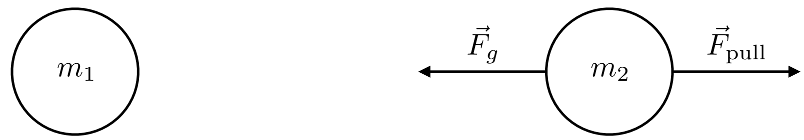
Let's attempt to address the question of how much energy it would take to blow up the Earth? Now, this isn't some fatalistic take on today's society, we want to determine the amount of energy it takes to completely pull apart every rock, every atom of Earth. Earth is held together through the gravitational force of its constituents, so to pull Earth apart, we need to do work against gravity to do this. Note that the range of the gravitational force is infinite: as long as the distance r between to massive objects is not infinite ($r < \infty$), then their gravitational force is non-zero. So, to completely blow up the Earth, we need to pull all of its atoms apart an infinite distance from one another. This is a really tall task, so we will simplify our picture of the Earth to analyze this.

Our model of the Earth will be the following: two masses m_1 and m_2 separated by distance r:



Okay, okay, so not very realistic. However, if r is about the radius of Earth and m_1 and m_2 are about half the mass of Earth, then by analyzing this system, we will be able to determine how much energy it would take to break Earth in two pieces. If we did that, I might claim victory.

Okay, to separate the masses, we need to do work against the force of gravity. What we will imagine doing is pulling mass m_2 from a separation of r with mass m_1 to a separation of ∞. We will just pull mass m_2 to the right to do this, which is opposite to the direction of gravitational force:



If we pull such that m_2 travels at a constant velocity, then $|\vec{F}_{\text{pull}}| = |\vec{F}_g|$, and we can determine how much work we would need to do to accomplish this. We do work from a distance r to a distance ∞, pulling in the direction of motion with a force equal in magnitude to the gravitational force. That is, the work we need to do to separate the masses is

$$\begin{aligned} W &= \int_r^\infty \vec{F}_{\text{pull}} \cdot d\vec{r}' = \int_r^\infty |\vec{F}_g|\, dr' = \int_r^\infty \frac{G_N m_1 m_2}{r'^2}\, dr' = -\frac{G_N m_1 m_2}{r'}\bigg|_r^\infty \\ &= \frac{G_N m_1 m_2}{r}\,. \end{aligned} \tag{6.15}$$

To do the integral of dx/x^2, note that the derivative of $1/x$ is

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}\, x^{-1} = -x^{-2}\,. \tag{6.16}$$

Therefore, the anti-derivative of x^{-2} is $-x^{-1}$. The work we had to do to separate the masses is still proportional to the product of their masses, but now only inversely proportional to their initial separation. The closer they are initially, the harder it is (the more work we have to do) to separate them.

Now, gravity is a conservative force as we discussed, so if we did this much work, then the opposite of this was initially stored as potential energy. That is, the gravitational potential energy of two masses m_1 and m_2 separated by distance r is

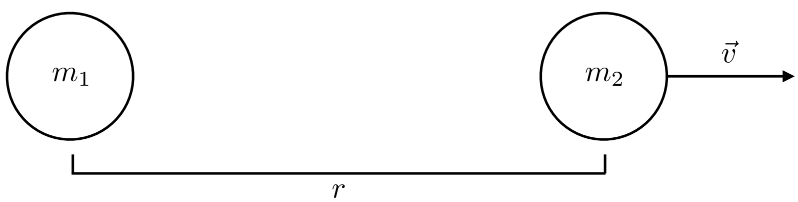
$$U = -\frac{G_N m_1 m_2}{r}\,. \tag{6.17}$$

Note the $-$ sign: it takes energy from us to separate the masses. With this gravitational potential energy, we can do everything we usually do with energy. And, remember, energy is a scalar (it has no direction), so it is easy to find total energies of multiple masses interacting gravitationally: we simply sum them up.

6.2.1 Escape Velocity

A related question to blowing up Earth is the following. How fast would you have to throw a ball upward, away from Earth, such that the ball ended up traveling an infinite distance from Earth? Within the context of our old model of gravity as a constant force, this was impossible, because constant force means eternally non-zero acceleration, so any finite velocity would eventually stop and reverse. For inverse-square gravity, we can travel fast enough to get out of the gravitational pull of Earth. We say that we have "escaped Earth's gravity" and the initial velocity needed to do this is called the **escape velocity**.

To determine the escape velocity, we are going back to our model of two masses m_1 and m_2. Now, however, we are going to give mass m_2 an initial velocity $\vec{v}$ pointed away from mass m_1:



How large must $|\vec{v}|$ be for m_2 to escape m_1's gravity? We can solve this with conservation of energy. Initially, the masses have a gravitational potential energy of

$$U_i = -\frac{G_N m_1 m_2}{r}\,, \tag{6.18}$$

and the mass m_2 has kinetic energy

$$K_i = \frac{1}{2} m_2 v^2\,. \tag{6.19}$$

The total initial energy of the system is then

$$E_{\text{tot}} = U_i + K_i = -\frac{G_N m_1 m_2}{r} + \frac{1}{2} m_2 v^2\,. \tag{6.20}$$

Now, when m_2 has escaped m_1, it is infinitely far away, so there is no gravitational potential energy, $U_f = 0$. Further, if mass m_2 juuuuust makes it out there, its final kinetic energy is 0 (no velocity). That is, the final total energy is 0, $E_{\text{tot}} = 0$. Setting initial and final energies equal, we have

$$-\frac{G_N m_1 m_2}{r} + \frac{1}{2} m_2 v^2 = 0\,, \tag{6.21}$$

or, solving for v, we find

$$v = \sqrt{\frac{2G_N m_1}{r}} \equiv v_{\text{esc}}\,, \tag{6.22}$$

where v_{esc} is the escape velocity. Setting the mass $m_1 = M_{\text{Earth}}$ and $r = R_{\text{Earth}}$, the escape velocity from the surface of the Earth is

$$v_{\text{esc}} = \sqrt{\frac{2G_N M_{\text{Earth}}}{R_{\text{Earth}}}} \approx \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \cdot 6 \times 10^{24}}{6 \times 10^6}} \text{ m/s} \approx 12 \text{ km/s} \approx 25,000 \text{ mph}\,. \tag{6.23}$$

Note that escape velocity depends on the initial distance from the gravitating object. What we derived above was the escape velocity from Earth's surface. If you are farther away from Earth when you start, then the velocity you need to escape Earth's gravity is correspondingly less. For example, the probes Voyagers 1 and 2 were launched from Earth in the 1970s with speeds much less than what would be needed to escape the gravitational force of the Sun, from a radius of the orbit of Earth. As they traveled through the solar system, they were able to get energy kicks from orbits around Jupiter, which pushed their velocities past the escape velocity of the Sun, at a radius of Jupiter's orbit. Voyagers 1 and 2 are just two of only five artificial objects that have attained solar escape velocity and have left the solar system.

6.2.2 Black Holes

For the last part of this chapter, let's throw this escape velocity idea on its head. While not a topic for this class, you might know that the speed of light in vacuum is an ultimate, universal speed limit. Nothing can travel faster than light. The speed of light is typically denoted as c and in SI units is

$$c = 3 \times 10^8 \text{ m/s}\,. \tag{6.24}$$

Imagine that there was a massive object whose escape velocity was $c = v_{\text{esc}}$. This would mean that not even light, traveling as fast as possible, could ever escape the gravitational pull of the object. I should also say that the manipulations I am going to do now are questionable, but give the right answer, so we will use them to provide insight into properties of such a massive body. Let's say this massive body has mass M and radius R, and we assume that its escape velocity from its surface (at radius R) is c. That is,

$$c = \sqrt{\frac{2G_N M}{R}} \,. \tag{6.25}$$

We can solve this instead for escape velocity c, for the radius R, where

$$R = \frac{2G_N M}{c^2} \,. \tag{6.26}$$

The interpretation of this distance is the following. If a mass M is entirely contained within a sphere of radius

$$R = \frac{2G_N M}{c^2} \,, \tag{6.27}$$

then the escape velocity from the surface of that sphere is the speed of light, c. As not even light can escape this mass, it is called a "black hole," a term introduced by John Wheeler.

This point should be emphasized. There is no black hole at the center of the Earth, because all of Earth's mass is not concentrated there. This radius for a given mass M is called its "Schwarzschild radius," after Karl Schwarzschild, a German physicist who first derived it, literally in the foxholes of World War I. For the Earth's mass of 6×10^{24} kg, its Schwarzschild radius would be

$$R_{\text{sch}} = \frac{2G_N M_{\text{Earth}}}{c^2} \approx 9 \text{ mm} \,. \tag{6.28}$$

That is, if all of Earth's mass were contained within a sphere of radius 9 mm, then it would form a black hole. You could hold it in your hand, but the gravitational force would be so strong you would quickly be sucked into it!

Chapter 7

Momentum

We will leave gravitation for now, and introduce another conservation law. Let's start with Newton's second law for a single object or particle of mass m:

$$\vec{F}_{\text{net}} = m\vec{a}\,. \tag{7.1}$$

Now, as a single entity, we are also going to imagine that the mass cannot change, that is, the object can't gain or lose mass. Parts of it can't fall off and nothing can stick to it. With this assumption, we can re-express Newton's second law as

$$\vec{F}_{\text{net}} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right) = \frac{d}{dt}(m\vec{v}) \equiv \frac{d\vec{p}}{dt}\,. \tag{7.2}$$

We call $\vec{p}$ the **momentum** of the object, but for now it is simply a placeholder name for the quantity $m\vec{v} = \vec{p}$.

7.1 Conservation of Momentum

Written in this form, however, Newton's second law has a nice interpretation: if $\vec{F}_{\text{net}} = 0$, then the time derivative of momentum $\vec{p}$ is 0, or that momentum does not change in time. That is, if $\vec{F}_{\text{net}} = 0$, then the particle's momentum is conserved. While this sounds profound, at this stage it is nothing more than the statement that if there are no forces, then there is no acceleration, so the particle's velocity is unchanged in time.

7.1.1 Impulse

Just like we did when introducing work and energy, we can anti-differentiate Newton's second law to determine the change in momentum from a force that acts over time. Recall that work is from a force that acts over a distance, while we call **impulse** the effect of force acting over time. We can integrate Newton's law over time:

$$\int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \int_{t_1}^{t_2} \vec{F}_{\rm net}\, dt = \Delta\vec{p}\,, \tag{7.3}$$

where $\Delta\vec{p}$ is the change in the momentum from time t_1 to time t_2:

$$\Delta\vec{p} = \vec{p}(t_2) - \vec{p}(t_1)\,. \tag{7.4}$$

We might call this the "impulse-momentum theorem," in analogy to the work-energy theorem, but that is not typically used.

One final point before moving on is that as we are currently considering a localized, isolated object, we imagine that it effectively has no extent. On a free-body diagram, it is but a point, so all of its mass m is localized at its position $\vec{x}$. The point at which mass is or can be integrated to be localized is called the center-of-mass of the system. We'll need this idea shortly.

7.1.2 Systems of Particles and Center-of-Mass

Okay, enough of one particle, let's imagine we have a system of two particles of mass m_1 and m_2 and we want to determine that system's dynamics with Newton's second law. The set-up is

and we give the masses some velocities $\vec{v}_1$ and $\vec{v}_2$. We can write down Newton's second law for each mass individually, where

$$\vec{F}_{\rm net,1} = \frac{d\vec{p}_1}{dt}\,, \qquad \vec{F}_{\rm net,2} = \frac{d\vec{p}_2}{dt}\,. \tag{7.5}$$

Note that the net forces on mass 1 may include forces exerted by mass 2, and vice-versa, if, for example, gravity is a relevant force. However, by Newton's third law, every force that 2 exerts on 1 has an equal and opposite partner of 1 exerted on 2. So, if we consider the total system of masses 1 and 2 together, then the only forces that affect the system are external to the two masses:

$$\vec{F}_{\rm ext} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{\rm net,1} + \vec{F}_{\rm net,2}\,. \tag{7.6}$$

Apparently, if there are no external forces, then the sum of the particle momenta is conserved. Forces only exerted between the particles do not affect the total momentum, by Newton's third law.

Let's keep going and attempt to interpret what the sum of momentum is. Again, assuming for simplicity that the individual particle masses are constant, we have

$$\vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 = \frac{d}{dt}(m_1\vec{x}_1 + m_2\vec{x}_2)\,. \tag{7.7}$$

Then, the term in parentheses is the mass-weighted position of the particles. Apparently, if there are no external forces, then there is no acceleration of this mass weighted position. Let's go a bit farther, multiplying and dividing by the total mass:

$$\vec{p}_1 + \vec{p}_2 = (m_1 + m_2)\frac{d}{dt}\left(\frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}\right)\,. \tag{7.8}$$

Now, the quantity on the right is called the **center-of-mass** and is the location at which all of the mass of the system can be imagined to be localized, for the purpose of where the external forces act. We denote this is

$$\vec{x}_{\rm cm} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}\,. \tag{7.9}$$

Note that if $m_1 \to 0$, then all of the mass is confined to be at mass 2, so $\vec{x}_{\rm cm} = \vec{x}_2$ (and similar if $m_2 = 0$). So, another way to express the sum of momentum of the two particles is:

$$\vec{p}_1 + \vec{p}_2 = (m_1 + m_2)\frac{d\vec{x}_{\rm cm}}{dt} = (m_1 + m_2)\vec{v}_{\rm cm}\,, \tag{7.10}$$

where $\vec{v}_{\rm cm}$ is the velocity of the center-of-mass. Then, Newton's second law can be re-expressed as

$$\vec{F}_{\text{ext}} = (m_1 + m_2)\vec{a}_{\text{cm}} \,, \tag{7.11}$$

where $\vec{a}_{\text{cm}}$ is the acceleration of the center-of-mass. That is, if there are no external forces, then the center-of-mass does not accelerate.

For a system of many particles, the story is the same, we just need to sum over all of their individual momenta. In that case, Newton's second law for a system of n particles is

$$\vec{F}_{\text{ext}} = \vec{a}_{\text{cm}} \sum_{i=1}^{n} m_i = \frac{d}{dt}\left(\sum_{i=1}^{n} \vec{p}_i\right) \,, \tag{7.12}$$

where

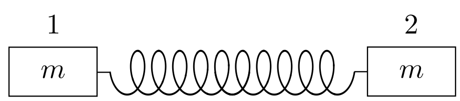
$$\sum_{i=1}^{n} m_i = m_1 + m_2 + \cdots + m_n \,, \qquad \sum_{i=1}^{n} \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \cdots \vec{p}_n \,. \tag{7.13}$$

We believe that there is nothing external to our universe, so necessarily there are no net external forces on our universe, $\vec{F}_{\text{ext,universe}} = 0$. This then implies that the sum of the momenta of all particles in the universe is conserved, unchanging in time.

Now, to say that the center-of-mass does not accelerate does not mean that individual particles cannot move when there are no external forces. For example, let's consider a couple different configurations of particles. First, let's assume that masses m_1 and m_2 are identical and equal to m. What is the velocity of the center-of-mass if $\vec{x}_1(t) = -\vec{x}_2(t)$, for all t? Well, we find this by simply plugging this into the expression for the center-of-mass:

$$\vec{x}_{\text{cm}} = \frac{m\vec{x}_1 + m\vec{x}_2}{m+m} = \frac{1}{2}\left(\vec{x}_1 + (-\vec{x}_1)\right) = 0 \,. \tag{7.14}$$

The masses can move, but the center-of-mass does not. An example of such a system would be two masses connected by a (massless) spring:



We can compress the spring and the masses will just oscillate back and forth, with $\vec{x}_1(t) = -\vec{x}_2(t)$, but won't be drifting anywhere in space. Correspondingly, if the center-of-mass doesn't move, then the net momentum is 0.

7.1.3 Spatial Translation Symmetry

While this configuration had no net momentum of the center-of-mass, we had discussed some time ago that there is no such thing as absolute velocity, so we could imagine this mass-spring system moving by at constant velocity and there still be no net forces. However, now in this case there would be a non-zero momentum of the system because the center-of-mass moves. This observation connects to the Noether's theorem interpretation of momentum.

If there are no external forces on our system, then the net momentum of the system is conserved. By Noether's theorem, if momentum is conserved, then there should be a corresponding symmetry under which the system of objects/particles is unchanged. As discussed, the net momentum of a system is intimately related to the motion of its center-of-mass, $\vec{x}_{\rm cm}$. The center-of-mass is some position in space and to move the center-of-mass requires **spatial translation**. For example, if the center-of-mass is initially at $\vec{x}_{\rm cm} = (1\text{ m})\hat{\imath}$ and we want to move it to $\vec{x}_{\rm cm} = (2\text{ m})\hat{\imath}$, then we need to translate one meter to the right.

We know how to do this translation: we simply give the system a non-zero momentum and the center-of-mass will move. The statement of conservation of momentum means that, by Noether's theorem, we can more the center-of-mass anywhere, and the physics of the system is unchanged. That is, if our system is invariant (= unchanged) to any spatial translation, then total momentum is conserved. This is, correspondingly, Noether's theorem for spatial translations.

Momentum conservation, like energy conservation, is typically much easier and more useful to directly use than Newton's second law to analyze the dynamics of a system. When only internal forces are relevant, i.e., force between objects in the system and no forces from the outside, momentum is conserved, and this is typically what is relevant for collisions analysis. Indeed, in my research which studies the collisions of protons at high energies, momentum conservation is extremely important for constraining the physics that may have been produced. We'll discuss more about collisions soon.

We argued that if energy is conserved, the laws of physics are independent of time. Correspondingly, if momentum is conserved then the laws of physics are independent of spatial position. Independence of time or position means that the derivative of the laws of physics with respect to these quantities is zero. Let's denote the laws of physics compactly as S. Conservation of energy means that

$$\frac{d}{dt}S = 0\,, \tag{7.15}$$

and conservation of the momentum vector means that

$$\frac{d}{dx}S = \frac{d}{dy}S = \frac{d}{dz}S = 0\,. \tag{7.16}$$

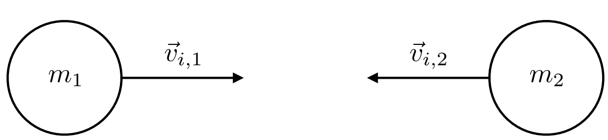
"Laws of physics" isn't just a simple function, as S must encode the motion and interactions of all particles in the universe. As such, S is a function of all the particles' trajectories, which are themselves functions of t, x, y, and z. We refer to S as the *action* and it is not just a function, but a *functional*. The statement that, for example, $dS/dt = 0$ means that the value of the action, as encoding the laws of physics, is independent of when in time all particle trajectories are measured from.

7.2 Collisions

We've now introduced conservation laws for energy and momentum, which, by Noether's theorem, correspond to symmetries under temporal and spatial translations, respectively. These conservation laws are especially useful for analyzing collisions to two objects, such as two balls, cars, bouncing a ball on the ground, an asteroid hitting a planet, two galaxies colliding, etc. Sorry for getting a bit carried away, but collisions are a very general physical phenomena, occurring essentially in any imaginable physical system. It's amazing that the very simple, yet enormously profound, ideas of conservation of energy and momentum are basically all we need to completely analyze any collision. Further, only using the principles of conservation of energy and momentum to analyze any collision is an extremely strong test of those fundamental ideas.

7.2.1 Elastic and Inelastic Collisions

So, with that prologue, it is useful for us to determine a taxonomy of different types of collisions, based on the relevant conservation laws. To set up, we will imagine colliding two objects of masses m_1 and m_2 with initial velocities $\vec{v}_{i,1}$ and $\vec{v}_{i,2}$



Then, the masses collide, and after the collision, the masses have velocities $\vec{v}_{f,1}$ and $\vec{v}_{f,2}$

First, as we have just discussed, if only forces internal to the mass 1–mass 2 system act on the masses, then momentum is conserved:

$$\vec{F}_{\text{ext}} = 0 \qquad \Rightarrow \qquad m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2} = m_1\vec{v}_{f,1} + m_2\vec{v}_{f,2}\,. \tag{7.17}$$

Energy is always conserved, but the expression of energy might be useless or irrelevant for the collision of the masses (sound, heat, etc.). So, constraining to useful forms of energy, potential and kinetic, kinetic energy is conserved in the collision if only conservative forces act on the masses:

$$\vec{F}_{\text{non-cons}} = 0 \qquad \Rightarrow \qquad \frac{1}{2}m_1v_{i,1}^2 + \frac{1}{2}m_2v_{i,2}^2 = \frac{1}{2}m_1v_{f,1}^2 + \frac{1}{2}m_2v_{f,2}^2\,. \tag{7.18}$$

Now, we can consider the four possible force combinations and give names to different types of collisions:

	$\vec{F}_{\text{non-cons}} = 0$	$\vec{F}_{\text{non-cons}} \neq 0$
$\vec{F}_{\text{ext}} = 0$	Momentum & Kinetic Energy Conserved “Elastic Collision”	Momentum Conserved “Inelastic Collision”
$\vec{F}_{\text{ext}} \neq 0$	Kinetic Energy Conserved	Neither Momentum nor Kinetic Energy Conserved

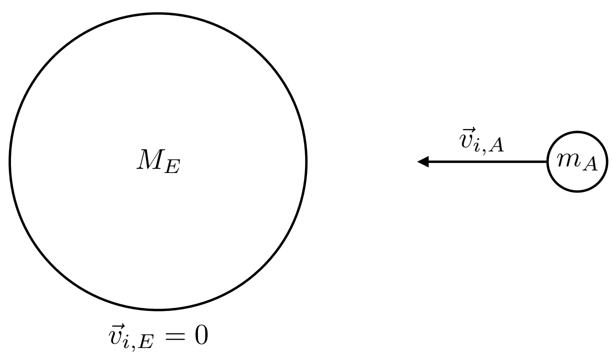
For analyzing a collision system, it just requires you to determine the relevant forces of the system to determine which conservation laws to use. We will mostly focus on **elastic** and **inelastic** collisions in this part of the course, as we are currently interested in momentum conservation. However, with Newton's second law or just conservation of energy, we had analyzed collisions of the $\vec{F}_{\text{ext}} \neq 0$ type in previous lectures. For example, if friction is a relevant force, it is external to the colliding objects and non-conservative, but we know how to deal with it.

7.2.2 K-T Extinction Event

Let's now take these ideas and analyze a particularly profound collision event in the history of Earth: the Cretaceous-Paleogene extinction event or the Cretaceous-Tertiary extinction

(K-T) that eliminated more than about 75% of extant life on Earth. The current widely accepted theory[1] for this mass extinction event was the impact of an asteroid at what is now the Yucatán peninsula about 66 million years ago, called the Chicxulub crater.

We will use the ideas of momentum conservation, energy conservation, and our theory of gravitation to determine the energy released by the asteroid's impact on Earth. Initially, before impact, the Earth and asteroid are in space as such:



We will assume that the Earth is at rest with respect to the space in which the collision occurs (the solar system). This is of course not strictly true, because the Earth is orbiting the sun, but we can imagine that the velocity of the asteroid is perpendicular to the velocity of Earth (i.e., radial toward the sun), so Earth's velocity is not relevant for the collision.

With this set-up, the initial momentum of the Earth-asteroid system is

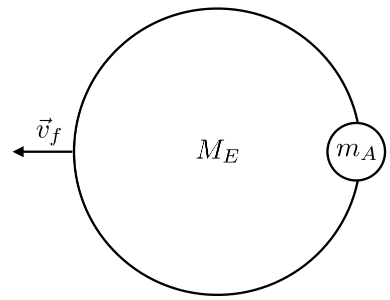
$$\vec{p}_i = \vec{p}_{i,E} + \vec{p}_{i,A} = m_A \vec{v}_{i,A} \,, \tag{7.19}$$

and the initial kinetic energy is

$$K_i = K_{i,E} + K_{i,A} = \frac{1}{2} m_A v_{i,A}^2 \,. \tag{7.20}$$

Now, what happens after the collision? The asteroid becomes embedded in the Earth, as such

[1]Luis W. Alvarez et al., "Extraterrestrial Cause for the Cretaceous-Tertiary Extinction", *Science* **208**, 1095-1108 (1980); Smit, J., Hertogen, J., "An extraterrestrial event at the Cretaceous-Tertiary boundary", *Nature* **285**, 198-200 (1980).



and now the Earth and asteroid travel with a common velocity $\vec{v}_f$.

Because the asteroid is stuck to Earth, there must be some "sticky" force responsible for attaching the asteroid to the Earth. This sticky force is entirely internal to Earth and the asteroid, and there are no relevant external forces present, so, by our taxonomy, momentum of the Earth-asteroid system is conserved. That is, the final momentum is

$$\vec{p}_f = \vec{p}_{f,E} + \vec{p}_{f,A} = (m_E + m_A)\vec{v}_f = m_A\vec{v}_i\,, \tag{7.21}$$

where we have simply called $\vec{v}_{i,A} \equiv \vec{v}_i$.

In contrast with momentum conservation, the sticky force is not conservative, as there is no well-defined potential energy ascribable to the sticky force. This is exactly analogous to the real, familiar force of sticky tape: it is definitely a force as it can accelerate objects or support them against gravity, but has no potential energy, in the same way that friction does not. Therefore, kinetic energy is not conserved in this collision. However, it will be useful later to determine how much kinetic energy was lost, so we can evaluate the final kinetic energy as

$$K_f = K_{f,E} + K_{f,A} = \frac{1}{2}(m_E + m_A)v_f^2\,. \tag{7.22}$$

To continue, we will assume that the collision only happens in one dimension, so we can drop the vectors in conservation of momentum. Then, the final velocity of the Earth-asteroid system is

$$(m_E + m_A)v_f = m_A v_i \qquad \Rightarrow \qquad v_f = \frac{m_A}{m_E + m_A}\,v_i\,. \tag{7.23}$$

Then, the final kinetic energy of the Earth-Asteroid system is

$$K_f = \frac{1}{2}(m_E + m_A)v_f^2 = \frac{1}{2}\frac{m_A^2}{m_E + m_A}\,v_i^2\,. \tag{7.24}$$

From one perspective, we are done: given the masses of the Earth and the asteroid and the initial velocity of the asteroid, we can determine the final velocity of the system. However, I want to go further. Because the "sticky" force that embeds the asteroid in Earth is non-conservative, kinetic energy from the asteroid initially is lost in the collision, as heat, explosion of rock, deafening sound, etc. So, how much energy is released in this asteroid collision?

The energy released in the collision is simply the difference between the initial and final kinetic energies of the system:

$$E_{\text{rel}} = K_i - K_f = \frac{1}{2}m_a v_i^2 - \frac{1}{2}\frac{m_A^2}{m_A + m_E}\, v_i^2 = \frac{1}{2}\frac{m_E m_A}{m_E + m_A}\, v_i^2\,. \tag{7.25}$$

The mass factor that appears in the final expression for the energy released is called the **reduced mass** and is denoted as

$$\mu \equiv \frac{m_E m_A}{m_E + m_A}\,. \tag{7.26}$$

So, we can equivalently write the released energy as

$$E_{\text{rel}} = \frac{1}{2}\frac{m_E m_A}{m_E + m_A}\, v_i^2 = \frac{m_E}{m_E + m_A}\cdot\frac{1}{2}m_A v_i^2 = \frac{m_E}{m_E + m_A}\, K_i\,, \tag{7.27}$$

where K_i is the initial kinetic energy of the asteroid.

What could this kinetic energy be, or rather, where did the astroid get this kinetic energy from? From the gravitational force of the Sun on the asteroid! To estimate the kinetic energy of the asteroid right before it hit Earth, let's imagine that it started from rest very far away from the Sun. Then, its initial energy would be 0: no kinetic energy and no gravitational potential energy. However, right before it hit Earth, it would have gained kinetic energy from losing gravitational potential energy by getting closer to the Sun. Its total energy must still be 0, and so

$$U_g + K_i = 0 = -\frac{G_N M_\odot m_A}{r_{E\odot}} + \frac{1}{2}m_A v_i^2\,, \tag{7.28}$$

or that the initial kinetic energy is

$$K_i = \frac{1}{2}m_A v_i^2 = \frac{G_N M_\odot m_A}{r_{E\odot}}\,, \tag{7.29}$$

where $M_\odot$ is the mass of the sun and $r_{E\odot}$ is the radius of Earth's orbit around the Sun. So, the energy released by the asteroid hitting Earth is

$$E_{\text{rel}} = \frac{m_E}{m_E + m_A} \frac{G_N M_\odot m_A}{r_{E\odot}} \,. \tag{7.30}$$

Now, we need to estimate the mass of the asteroid, m_A. It is estimated that the diameter of the asteroid, inferred from the Chicxulub crater's size, was about 10 km. This is about a factor of 1000 times smaller than the size of the Earth. The amount of mass (= stuff) in an object is determined by its volume, which is proportional to the cube of the diameter. So, if the asteroid has a diameter that is a thousand times smaller than Earth, then its volume is smaller by the cube of this, or a factor of one billion. So, we estimate the mass of the asteroid to be a billion times smaller than the mass of the Earth,

$$m_A \approx 10^{-9} m_E \,. \tag{7.31}$$

With this identification, the ratio factor in front of the released energy is very nearly just 1:

$$\frac{m_E}{m_E + m_A} = \frac{1}{1 + \frac{m_A}{m_E}} \approx \frac{1}{1 + 10^{-9}} = 1 - 10^{-9} + \cdots \,, \tag{7.32}$$

using the binomial expansion. So, for our estimate, we will just set it to 1.

The energy released by the asteroid is then

$$E_{\text{rel}} \approx \frac{G_N M_\odot m_A}{r_{E\odot}} \,, \tag{7.33}$$

where the values of the factors in this expression are:

$$G_N = 6.67 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2} \,, \qquad M_\odot = 2 \times 10^{30} \text{ kg} \,, \tag{7.34}$$

$$m_A \approx 10^{-9} m_E \approx 6 \times 10^{15} \text{ kg} \,, \qquad r_{E\odot} = 1.5 \times 10^{11} \text{ m} \,. \tag{7.35}$$

Plugging in these numbers, we find that the energy released in the asteroid collision to be

$$E_{\text{rel}} \approx 5 \times 10^{24} \text{ J} \,. \tag{7.36}$$

This is an exceedingly large amount of energy, enough to wipe out most life on Earth. It is estimated that Mt. St. Helens released about 10^{18} J of energy in its explosion in 1980. The asteroid that created the Chicxulub crater would have been like a million Mt. St. Helens events simultaneously.

7.3 Conservation Laws in Multiple Dimensions

We will continue our discussion of momentum conservation, focusing on momentum conservation in multiple dimensions. Again, for a system of n particles or objects, we can write Newton's second law in terms of momentum as

$$\vec{F}_{\text{ext}} = \frac{d}{dt}\sum_{i=1}^{n}\vec{p}_i = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n\right)\,, \tag{7.37}$$

where $\vec{p}_i$ is the momentum of particle i and $\vec{F}_{\text{ext}}$ is the sum of forces external to the n particles. If there is no relevant net external force, then the time derivative of the sum of momentum is 0:

$$\vec{F}_{\text{ext}} = 0 = \frac{d}{dt}\sum_{i=1}^{n}\vec{p}_i\,, \tag{7.38}$$

or that the sum of momentum is conserved, unchanged, in time:

$$\sum_{i=1}^{n}\vec{p}_i = \text{constant}\,. \tag{7.39}$$

This is a vector equation, so the sum of each component is conserved individually if $\vec{F}_{\text{ext}} = 0$:

$$\sum_{i=1}^{n}\vec{p}_{i,z} = \text{constant}\,, \qquad \sum_{i=1}^{n}\vec{p}_{i,y} = \text{constant}\,, \qquad \sum_{i=1}^{n}\vec{p}_{i,z} = \text{constant}\,, \tag{7.40}$$

where $p_{i,x}$, $p_{i,y}$, and $p_{i,z}$ are the x-, y-, and z- components of the momentum of particle i, respectively. We had previously exploited momentum conservation to analyze collisions that exclusively occurred in one dimension; in this lecture, we will study the constraints that momentum conservation in two dimensions imposes on systems with multiple particles. We will focus on two dimensions rather than three dimensions because going to three dimensions mostly just adds complication and additional bookkeeping.

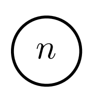
7.3.1 Simple Model of Neutron Decay

To set the stage for this lecture, I'd like to study the physics of the Reed reactor. How many of you work at the reactor, or are in training? How many of you even applied to Reed in part because of the reactor? How many of you are familiar with the reactor at all? Like any nuclear reactor, the Reed reactor is powered at its core by a radioactive element that decays to other particles after a characteristic time called the *half-life*. The Reed reactor uses plutonium as its core, but the amount of plutonium is tiny and is much too small to output a useful amount of energy, or to ever be a threat to campus. Nevertheless, there is very interesting physics in nuclear decays, and in this lecture, we will explore one aspect through momentum conservation.

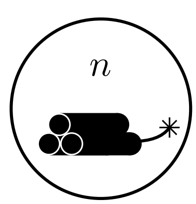
A complete, or even simply honest, discussion of nuclear decay requires concepts of both special relativity and quantum mechanics, but we'll be able to simplify our discussion enough to not need their details. Fundamentally, nuclear decay is a consequence of the decay of the neutron, one of the constituents of atomic nuclei, along with protons. A neutron is observed to decay to a proton and an electron, and we denote this decay as

$$n \to p^+ + e^- \,, \tag{7.41}$$

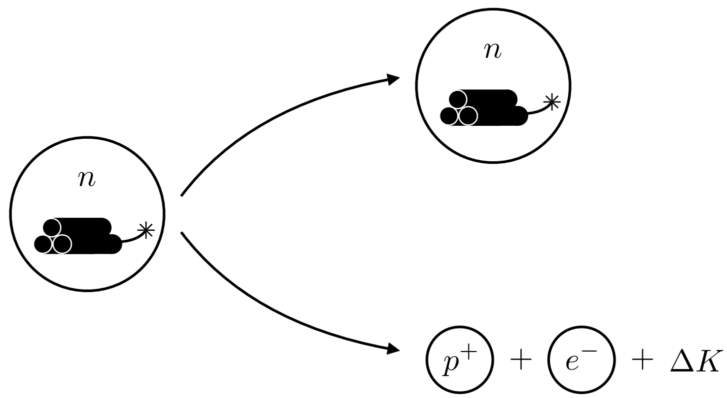
where we denote the neutron as n, the proton as p^+, and the electron is e^-. The superscripts denote the electric charge of the proton and electron; the neutron is electrically neutral. You'll be introduced to these concepts in the course next semester. In addition to the neutron producing a proton and an electron, kinetic energy is also produced that pushes the proton and electron apart. Let's call this kinetic energy ΔK. So, our model for neutron decay is the following. We initially have a neutron hanging out in space at rest, $\vec{v} = 0$:



As a model for decay and kinetic energy release, we imagine that there is a little time bomb in the neutron:



After the half-life of the neutron elapses, about ten minutes, there is a 50% chance that the neutron decays and a 50% chance that it does not:



If the neutron decayed, then this model/hypothesis for neutron decays makes concrete predictions for the kinetic energy of the electron that we can test in experiment.

If the neutron is at rest when it decays, then it has no momentum. The decay/explosion of the neutron into a proton and electron exclusively involves forces internal to the proton/electron system, so momentum is conserved. This means that the sum of proton and electron momentum is 0:

$$m_p\vec{v}_p + m_e\vec{v}_e = 0\,. \tag{7.42}$$

As mentioned earlier, we will study this decay in two dimensions, so we can write this out in components as

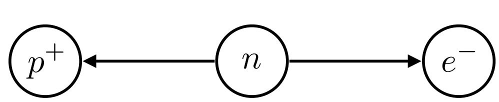
$$m_pv_{p,x} + m_ev_{e,x} = 0\,, \qquad\qquad m_pv_{p,y} + m_ev_{e,y} = 0\,. \tag{7.43}$$

Further, the neutron decay produces a kinetic energy of ΔK, carried by the proton and electron. This means that

$$\Delta K = \frac{1}{2}m_pv_p^2 + \frac{1}{2}m_ev_e^2 = \frac{1}{2}m_p\left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2}m_e\left(v_{e,x}^2 + v_{e,y}^2\right)\,, \tag{7.44}$$

where we have written the kinetic energy out in components on the right. Now, we would like to massage these equations to determine the energy of the electron individually.

The first thing that we note is that, although we have expressed momentum and energy with both components of the proton's and electron's velocity vectors, we can set one of the components to 0. Momentum conservation requires that the velocities of the proton and electron are back-to-back



So, we can just orient our axes such that there is no y-component of velocity: $v_{p,y} = v_{e,y} = 0$. With this orientation, the x-component of velocity is simply the total speed of the proton and electron:

$$v_{p,x} = v_p \,, \qquad\qquad v_{e,x} = v_e \,. \tag{7.45}$$

Then, conservation of momentum and energy are

$$m_p v_p + m_e v_e = 0 \,, \qquad\qquad \Delta K = \frac{1}{2}m_p v_p^2 + \frac{1}{2}m_e v_e^2 \,. \tag{7.46}$$

Momentum conservation implies that

$$v_p = -\frac{m_e}{m_p} v_e \,, \tag{7.47}$$

and plugging this into energy conservation, we find

$$\Delta K = \frac{1}{2}m_p\left(\frac{m_e}{m_p}v_e\right)^2 + \frac{1}{2}m_e v_e^2 = \left(\frac{m_e+m_p}{m_p}\right)\cdot\frac{1}{2}m_e v_e^2 = \frac{m_e+m_p}{m_p}\,K_e \,. \tag{7.48}$$

Solving for the kinetic energy of the electron, we find

$$K_e = \frac{1}{2}m_e v_e^2 = \frac{m_p}{m_p+m_e}\,\Delta K \,. \tag{7.49}$$

The mass of the proton is about 2000 times that of the electron, so we can approximate the mass ratio factor as

$$\frac{m_p}{m_p+m_e} = \frac{1}{1+\frac{m_e}{m_p}} = \left(1+\frac{m_e}{m_p}\right)^{-1} \approx 1 - \frac{m_e}{m_p} \approx 1 - \frac{1}{2000} \approx 1 \,. \tag{7.50}$$

So, the kinetic energy of the electron would just be approximately the kinetic energy released by the neutron decay:

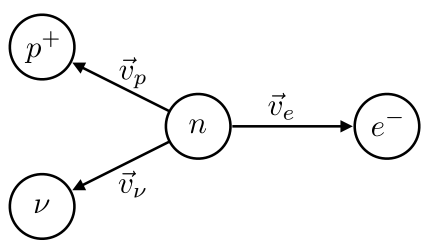
$$K_e \approx \Delta K \,. \tag{7.51}$$

ΔK is a fixed value of about 1.2×10^{-13} J or 780 keV (that is, kilo-electron Volts), so this model predicts that the kinetic energy of every electron produced from neutron decay always carries this kinetic energy. So, we test this prediction by preparing a large number of neutrons, let them decay, and measure the kinetic energy of the produced electron from decay. If we always see the electron carry $K_e \approx \Delta K = 1.2 \times 10^{-13}$ J, then we gain evidence for the model of $n \to p^+ + e^-$ decay.

If one does this experiment, this is not what is found! Instead of the electron always carrying kinetic energy ΔK, it is found that the electron carries kinetic energy $K_e \in [0, \Delta K]$, bounded from above by ΔK. So our hypothesis was incorrect. What is the simplest thing we can do to modify it? We could throw out momentum and energy conservation, but that is very dramatic because we have so much evidence for their conservation otherwise. By Occam's razor, the simplest explanation is typically the correct one, so we don't want to consider non-conservation unless we are absolutely forced to. Well, in the decay of the neutron, we *observe* the proton and the electron decay products. However, what if there was another, third, decay product that we could no observe directly? How would that affect the kinetic energy and momentum that the electron carried?

Let's now hypothesize that the neutron decays to three particles, referred as a three-body decay, $n \to p^+ + e^- + \nu$, where the third particle is denoted with the Greek letter nu, ν, and is called a *neutrino* (story to follow). What do conservation of momentum and energy look like now?

We illustrate the decay of the neutron now as



with conservation of momentum and energy:

$$m_e\vec{v}_e + m_p\vec{v}_p + m_\nu\vec{v}_\nu = 0\,, \qquad \Delta K = \frac{1}{2}m_e v_e^2 + \frac{1}{2}m_p v_p^2 + \frac{1}{2}m_\nu v_\nu^2\,. \tag{7.52}$$

(I should say that these are not the correct conservation laws for this decay because the neutrino is traveling at essentially the speed of light. However, it is sufficient to illustrate the interesting physics.)

Unlike the two-body decay we had studied earlier, this three-body decay is not confined to a line, so we have to consider a more general, two-dimensional, conservation law. We can, however, still rotate axes to be convenient; we will choose to align the electron velocity with the x-axis. Then the conservation laws in components are

$$m_e v_e + m_p v_{p,x} + m_\nu v_{\nu,x} = 0\,, \qquad m_p v_{p,y} + m_\nu v_{\nu,y} = 0\,, \tag{7.53}$$
$$\Delta K = \frac{1}{2}m_e v_e^2 + \frac{1}{2}m_p\left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2}m_\nu\left(v_{\nu,x}^2 + v_{\nu,y}^2\right)\,.$$

Now, we have three conservation laws but five unknowns $(v_e, v_{p,x}, v_{p,y}, v_{\nu,x}, v_{\nu,y})$, so we cannot uniquely solve the systems of equations. However, we can simplify it and eliminate dependence on the neutrino velocity, $\vec v_\nu$. From the conservation laws for momentum, we have

$$v_{\nu,x} = -\frac{m_e v_e + m_p v_{p,x}}{m_\nu}\,, \qquad v_{\nu,y} = -\frac{m_p}{m_\nu}v_{p,y}\,. \tag{7.54}$$

Plugging these expressions into the conservation of energy, we find

$$\Delta K = frac12 m_e v_e^2 + \frac{1}{2}m_p\left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2}m_\nu\left(\frac{(m_e v_e + m_p v_{p,x})^2}{m_\nu^2} + \frac{m_p^2}{m_\nu^2}v_{p,y}^2\right) \tag{7.55}$$
$$= \frac{1}{2}m_e v_e^2\left(1 + \frac{m_e}{m_\nu} + \frac{m_p}{m_\nu}\frac{v_{p,x}}{v_e}\right) + \frac{1}{2}m_p v_p^2\left(1 + \frac{m_p}{m_\nu}\right)\,,$$

which has no residual dependence on the unobservable neutrino velocity. We can solve for $K_e = \frac{1}{2}m_e v_e^2$ and find

$$K_e = \frac{1}{2}m_e v_e^2 = \frac{\Delta K - K_p\left(1 + \frac{m_p}{m_\nu}\right)}{1 + \frac{m_e}{m_\nu} + \frac{m_p}{m_\nu}\frac{v_{p,x}}{v_e}}\,, \tag{7.56}$$

where $K_p = \frac{1}{2}m_p v_p^2$.

Now, unlike the case when we hypothesized that the neutron decayed exclusively to the proton and electron, the kinetic energy of the electron has a range of possible values, just as we measure in experiment. For example, the speed of the electron can vanish: $v_e = 0$, and momentum and energy still be conserved through the proton and neutrino. Thus, conservation laws can be exploited to determine and identify new particles you can otherwise not directly observe!

This conservation of momentum and energy (and angular momentum) argument for neutron decay, or radioactive decay more generally, was used in the early 1930s to postulate the neutrino. In the 19-teens, 20s, and 30s, people like the Curies and Lise Meitner identified radioactivity in heavy, unstable isotopes, and even constructed a theory for their mechanism.

However, it was observed that the conservation laws had problems connecting theory to experimental data. To rectify it, Wolfgang Pauli postulated the existence of a new particle that was also produced in radioactive decay which he called the "neutron," but we now call the "neutrino" ("little neutral one" in Italian, coined by Enrico Fermi). Pauli had been invited to a conference on radioativity in Tübingen, Germany, in the early 1930s when he had the idea. Unfortunately, his attendance was required at a ball in Zürich, Switzerland, at the same time, so couldn't attend the conference. However, in lieu of attending, Pauli wrote a letter to Lise Meitner who was at the conference in which he laid out his idea for the "neutrino." Pauli addressed the attendees of the conference in the letter as:[2]

> Dear Radioactive Ladies and Gentlemen,

and further apologized for his absence!

7.4 Working with the Center-of-Mass

I hope I have impressed upon you the importance of momentum conservation and how it has very relevant consequences for our world. In this lecture, we're going to start pivoting from momentum conservation to the discussion of dynamics of rotations. Our first step will be, as always, to revisit Newton's second law written in a form we had introduced previously. For a system of n masses, we had shown that Newton's second law could be written as

$$\vec{F}_{\text{ext}} = \left(\sum_{i=1}^{n} m_i\right) \frac{d^2\vec{x}_{\text{cm}}}{dt^2}\,, \tag{7.57}$$

where

$$\sum_{i=1}^{n} m_i = m_1 + m_2 + \cdots + m_n\,, \tag{7.58}$$

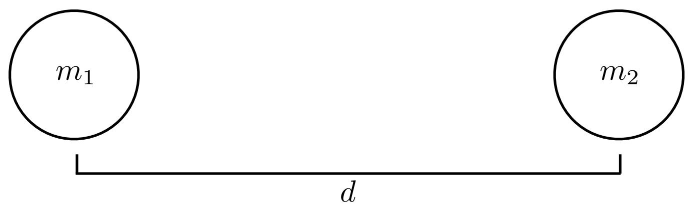
is the sum of masses in the system and $\vec{x}_{\text{cm}}$ is the position of the center-of-mass

$$\vec{x}_{\text{cm}} = \frac{\sum_{i=1}^{n} m_i\vec{x}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2 + \cdots + m_n\vec{x}_n}{m_1 + m_2 + \cdots + m_n}\,. \tag{7.59}$$

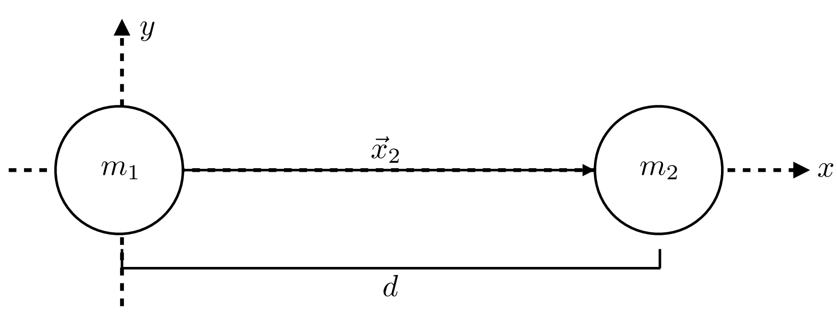
[2] W. Pauli,"Dear radioactive ladies and gentlemen," from a letter to Lise Meitner, dated Dec. 1930 [reprinted in Phys. Today **31**, no. 9, 27 (1978)].

This has, so far, mostly been a notational convenience rather than a powerful predictive framework, but we had mentioned that two mutually-gravitationally bound masses orbit their common center-of-mass. Why is that the case and what consequences does the center-of-mass have for systems that in general are composed of many individual masses?

Let's first just study a system that consists of two masses, m_1 and m_2, separated by a distance d:



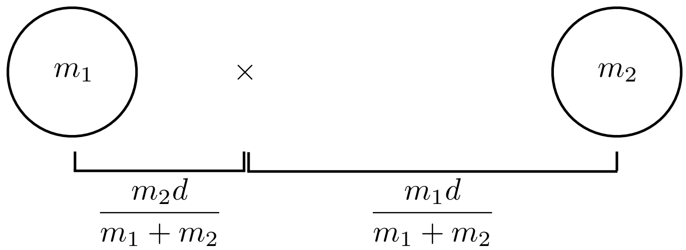
To calculate their center-of-mass, we need to know the position vectors of the masses which, in turn, requires setting up a coordinate system. As I have always said, life cannot imitate art in physics, which means that this coordinate system is simply a tool to be able to talk concretely about the masses, but their physical location, and hence the physical location of their center-of-mass, is the same place, regardless of the coordinates used. So, we simply pick convenient coordinates with, say, mass 1 located at the origin:



Then, the center-of-mass location in these coordinates is

$$\vec{x}_{\rm cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} = \frac{0 \cdot m_1 + m_2 d\,\hat{\imath}}{m_1 + m_2} = \frac{m_2 d}{m_1 + m_2}\,\hat{\imath}\,. \tag{7.60}$$

Note that m_1 and m_2 are positive, so $\frac{m_2}{m_1+m_2} \leq 1$, which means that the location of the center-of-mass lies between the masses:

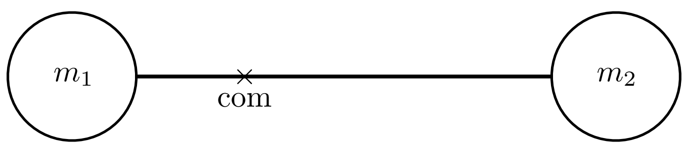


Of course, the sum of distances of each mass to their combined center-of-mass is d:

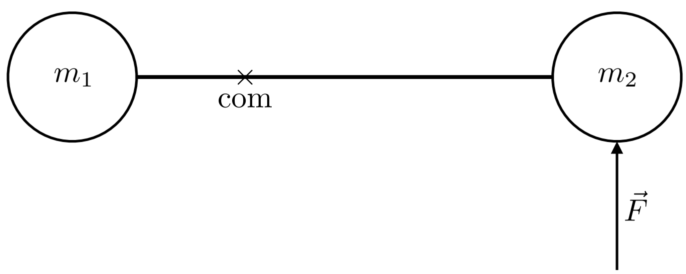
$$d = \frac{m_2 d}{m_1 + m_2} + \frac{m_1 d}{m_1 + m_2}\,. \tag{7.61}$$

7.4.1 Forces on Extended Objects

Now, let's imagine that the two masses are connected by a rigid, massless rod, like so

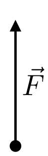


Now, with this set-up, let's imagine exerting a force upward on mass 2, like so



What does this mass-rod system do, immediately after the force is applied?

To answer this question, let's consider the Newton's law we derived for a system of masses. First, the free-body diagram for this system is



$$\Rightarrow \vec{F} = (m_1 + m_2)\vec{a}_{\rm cm} \tag{7.62}$$

where we imagine the force acting exclusively at the center-of-mass. So, the acceleration of the center-of-mass is

$$a_{\rm cm} = \frac{F}{m_1 + m_2} \text{ upward.} \tag{7.63}$$

Further, it doesn't matter where on this system we consider the force exerted, on mass 1, somewhere on the rod, etc., the center-of-mass *always* accelerates with this rate. What response the individual masses have depends on where the force is applied, but the center-of-mass just accelerates according to the net external force.

This can be exploited to great effect. First, let's consider putting this mass-rod system near the surface of the Earth. Now, there is a net external force, namely, gravity. By our work thus far, this force acts at the center-of-mass,

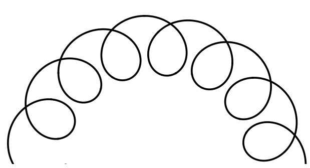
$$\vec{F}_g = -(m_1 + m_2)g\,\hat{\jmath} = (m_1 + m_2)\vec{a}_{\rm cm}\,, \qquad \text{or that} \qquad \vec{a}_{\rm cm} = -g\,\hat{\jmath}\,. \tag{7.64}$$

So, if we just let this thing fall, the center-of-mass would accelerate at g. Conversely, to hold the object up, we need to apply some normal force that prevents the center-of-mass from accelerating downward. As long as we do this, the object will not fall. So, we only need to hold it up by the center-of-mass.

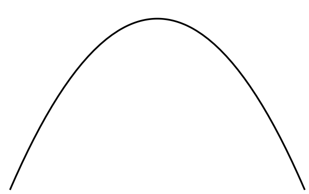
Example

To illustrate this phenomena, let's consider a croquet mallet, and on it, someone has nicely put LEDs at the location of its center-of-mass. I will turn off the lights and then throw the mallet across the room, ensuring that the head tumbles over the handle. What will the trajectory of the lights be?

(a)

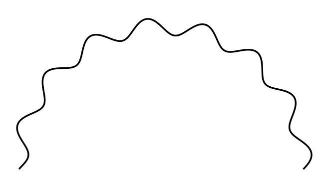


(b)



(c) Lights do not work when moving

(d)



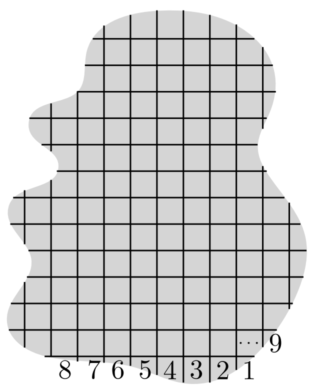
By Grabthar's hammer, let's try this out! (See `https://youtu.be/I4EkPHyW3ig`)

7.4.2 Center-of-Mass of an Extended Object

Now, this two-mass system is just the starting point; we would like to determine the center-of-mass for an arbitrary system of masses. We will consider some blob as the system of interest:

Let's assume that the blob is just two dimensional for simplicity. How do we calculate its center-of-mass?

As often in this game, let's break it up into many small masses and sum them up. So, we will consider



where the small submasses are labeled. The total mass M of the object is

$$M = m_1 + m_2 + \cdots = \sum_i m_i \,. \tag{7.65}$$

Further, let's introduce A_i as the area of the ith piece. Then we call the **density** σ_i the ratio of mass to area:

$$\sigma_i = \frac{m_i}{A_i} \,. \tag{7.66}$$

Now, this is two dimensional, so we can specify the point i by its x- and y-coordinates:

$$\sigma_i \equiv \sigma(x, y) = \frac{m(x, y)}{A(x, y)} \,. \tag{7.67}$$

If the areas are taken to be very small rectangles with sides of length dx and dy, the area is

$$A(x, y) = dx\, dy\,, \tag{7.68}$$

and for small area, mass is also small, so we denote $m(x, y) \equiv dm$. Schematically, we then have that the density is

$$\sigma(x, y) = \frac{dm}{dx\, dy}\,. \tag{7.69}$$

The total mass is simply the sum of all of these small masses. In the limit that the masses get infinitesimally small, the sum transforms into an integral

$$M = \sum_i m_i \qquad \to \qquad \int dm = \iint \frac{dm}{dx\, dy}\, dx\, dy = \iint \sigma(x, y)\, dx\, dy\,. \tag{7.70}$$

On the right, we have introduced a double integral over the density to calculate mass. A double integral is just two nested integrals: do the x integral first, assuming y is constant, then integrate over y. So, our expression for the mass of a complicated shape is

$$M = \iint \sigma(x, y)\, dx\, dy\,. \tag{7.71}$$

Now, to calculate the center-of-mass of this object, we weight each tiny mass by its position vector and divide by the total mas

$$\vec{x}_{\text{cm}} = \frac{1}{M} \int \vec{r}\, dm = \frac{1}{M} \iint \sigma(x, y)\, \vec{r}\, dx\, dy = \frac{1}{\iint \sigma(x, y)\, dx\, dy} \iint \sigma(x, y)\, (x\hat{\imath} + y\hat{\jmath})\, dx\, dy\,. \tag{7.72}$$

To integrate over the vector of position $\vec{r}$, the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are constants, unaffected by integration. They need to be kept around to determine the vector position of the center-of-mass.

Example

With these observations, we are going to think about its consequences for in an example. Consider an arbitrary shaped mass. I hang the mass from a nail and it swing freely until it comes to rest. With respect to the nail, where is the center-of-mass?

(a) Right of the nail

(b) Left of the nail

(c) Below the nail

(d) Above the nail

(See `https://youtu.be/0fFvrttovcU`) Can you think of a method for determining the exact position of the center-of-mass of an object using this result?

Chapter 8

Rotation

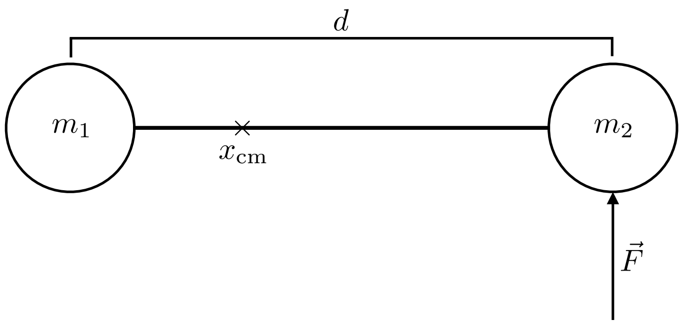
Previously, we demonstrated and discussed properties of the center-of-mass of an object that has extended or irregular structure. Regardless of what force is exerted on an object or how, the center-of-mass of that object accelerates simply according to Newton's second law,

$$\vec{F}_{\text{ext}} = M\frac{d^2\vec{x}_{\text{cm}}}{dt^2}\,, \tag{8.1}$$

where M is the total mass of the object. Also, as we observed previously, this clearly isn't the whole story. When I threw the mallet, you indeed saw that the center-of-mass traveled in a parabolic trajectory, as expected from projectile motion. However, as the center-of-mass was traveling as a projectile, the head and handle of the mallet rotated end-over-end about the center-of-mass, clearly a non-projectile motion. How do we model this motion and describe the more general motion of the entire mallet, not just its center-of-mass?

8.1 Rotation about the Center-of-Mass

Let's start this discussion with the physical set-up that we considered at the end of the previous chapter. Let's consider two masses separated by a distance d by a rigid rod and apply vertical force $\vec{F}$ to mass m_2:



From our earlier analysis, we know that applying this force accelerates the center-of-mass

$$\vec{F} = (m_1 + m_2)\vec{a}_{\rm cm} \qquad \text{or that} \qquad \vec{a}_{\rm cm} = \frac{F}{m_1+m_2}\hat{\jmath}\,. \tag{8.2}$$

So, the center-of-mass just accelerates vertically according to the magnitude of $\vec{F}$ and the sum of the masses. What about the accelerations of masses 1 and 2 individually?

8.1.1 Forces on Extended Objects

Let's break apart Newton's law as an explicit sum over the accelerations of masses 1 and 2:

$$\vec{F} = (m_1+m_2)\vec{a}_{\rm cm} = (m_1+m_2)\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2} = m_1\vec{a}_1 + m_2\vec{a}_2\,. \tag{8.3}$$

The sum of accelerations $\vec{a}_1$ and $\vec{a}_2$ is constrained by the acceleration of the center-of-mass, so let's express the individual accelerations as

$$\vec{a}_1 = \vec{a}_{\rm cm} + \Delta\vec{a}_1\,, \qquad\qquad \vec{a}_2 = \vec{a}_{\rm cm} + \Delta\vec{a}_2\,, \tag{8.4}$$

for some accelerations $\Delta\vec{a}_1, \Delta\vec{a}_2$. So far, this is just a tautology; we have done nothing but shift our notation. However, now we can find a simple relationship between $\Delta\vec{a}_1$ and $\Delta\vec{a}_2$ from Newton's second law:

$$\vec{F} = (m_1+m_2)\vec{a}_{\rm cm} = m_1(\vec{a}_{\rm cm} + \Delta\vec{a}_1) + m_2(\vec{a}_{\rm cm} + \Delta\vec{a}_2)\,, \tag{8.5}$$

or that $m_1\Delta\vec{a}_1 + m_2\Delta\vec{a}_2 = 0$. We'll use this relationship to relate $\Delta\vec{a}_1$ to $\Delta\vec{a}_2$, where

$$\Delta\vec{a}_1 = -\frac{m_2}{m_1}\Delta\vec{a}_2\,. \tag{8.6}$$

So, the individual accelerations are

$$\vec{a}_1 = \vec{a}_{\rm cm} - \frac{m_2}{m_1}\Delta\vec{a}_2\,, \qquad \vec{a}_2 = \vec{a}_{\rm cm} + \Delta\vec{a}_2\,. \tag{8.7}$$

There are more constraints we can exploit.

The fact that the masses are connected by a perfectly rigid rod constrains their motion about the center-of-mass and relative to one another. The center-of-mass from this applied force moves exclusively vertically. Because of the rigidity of the rod, mass 2, for example, remains a fixed distance away from the center-of-mass. However it can, in principle, have any orientation about the center-of-mass. So, what type of motion is constrained to be a fixed distance from a point, but have any orientation? Circular motion!

Therefore, masses 1 and 2 travel in a circular orbit about the center-of-mass as the center-of-mass accelerates vertically. So, somehow this circular motion is accounted for in the $\Delta\vec{a}_2$ acceleration we have yet to find. Again, let's use rigidity to find more constraints on this circular motion. Because the rod is perfectly rigid, the masses m_1 and m_2 are always on opposite sides of their respective circular orbits; that is, they necessarily orbit with the same angular velocity. If they had different angular velocities, then they would no longer be antipodal, and the rod must crumple, but that can't happen.

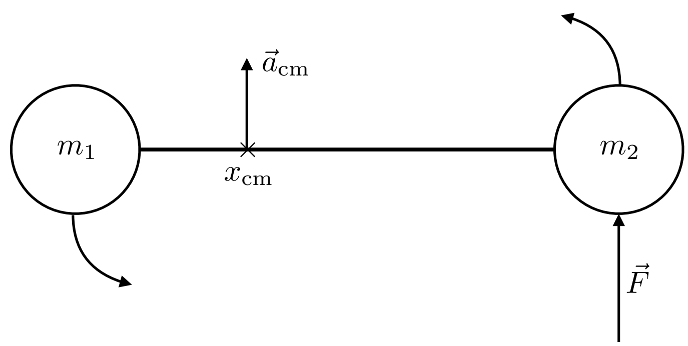
Angular velocity ω is defined as the first time derivative of the angle θ through which an object travels:

$$\omega = \frac{d\theta}{dt}\,. \tag{8.8}$$

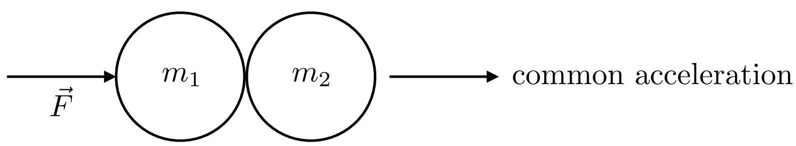
We can take a second derivative to identify the angular acceleration, α:

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}\,. \tag{8.9}$$

Now, if the angular velocities of masses 1 and 2 are identical, $\omega_1 = \omega_2$, then so too must their angular accelerations be: $\alpha_1 = \alpha_2$. The force $\vec{F}$ works to angularly accelerate the masses in the same direction of rotation:



So, we can imagine that $\vec{F}$ is responsible for the rotation of m_1 and m_2 in the counterclockwise direction, in the same way that it would accelerate them linearly if it pushes m_1 and m_2 when touching, i.e.,



To rotate m_1 and m_2 about the center-of-mass, then $\vec{F}$ has to push against the inertia of both masses, to get them both to rotate. Just like in the linear case where the two masses would have a common linear acceleration and the force would push against the combine mass of the two blocks,

$$\vec{F} = (m_1 + m_2)\vec{a}\,, \qquad \text{(linear)} \tag{8.10}$$

the fact that in this rotating case the two masses have the same angular acceleration suggests a nice way to interpret.

The force $\vec{F}$ provides a tangential acceleration of the two masses in their orbit about the center-of-mass. That is, Newton's second law implies

$$\vec{F} = m_2\vec{a}_{\tan,2}\,, \qquad \text{or} \qquad F = m_2 a_{\tan,2}\,, \tag{8.11}$$

where we can drop the vectors because rotation is occurring in one plane (the plane of the page). Note that only the acceleration of mass 2 appears because $\vec{F}$ only directly acts on mass 2. Tangential acceleration is related to angular acceleration by a factor of radius from the center of the orbit. What is this relevant radius R, to relate

$$a_{\tan,2} = R\alpha\,? \tag{8.12}$$

Right at the instant when F is applied, only mass 2 moves; mass 1 remains stationary (for an instant). So, at that moment, mass 2 is orbiting mass 1, a distance d away. Therefore, we initially have

$$a_{\tan,2} = d\alpha\,, \tag{8.13}$$

so that Newton's law is simply $F = m_2 d\alpha$. Now, one can use this to go back and solve for the unknown $\Delta\vec{\alpha}_2$, but we won't do that here (though I encourage you to do so!). I want to massage this expression into another form that exclusively uses information about the rotation of the masses about their common center-of-mass.

Note the string of identities:

$$\begin{aligned} F\frac{m_1 d}{m_1+m_2} &= \frac{m_1 m_2 d}{m_1+m_2}\,d\alpha = \left(m_1\frac{m_2^2 d^2}{(m_1+m_2)^2} + m_2\frac{m_1^2 d^2}{(m_1+m_2)^2}\right)\alpha = \left(m_1 R_1^2 + m_2 R_2^2\right)\alpha \\ &= F R_2\,. \end{aligned} \tag{8.14}$$

Here, R_1 and R_2 are the distances from the center-of-mass to the respective masses. The factor of R_2 that multiplies F is the distance from the location of the applied force $\vec{F}$ to the center-of-mass.

I emphasize that this is nothing more than Newton's second law, but expressed in a way useful for rotations. The mass-times-radius factors are called **moments of inertia** I defined as

$$I = MR^2\,, \tag{8.15}$$

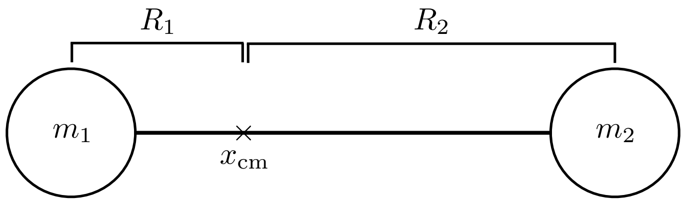
where M is the mass of the object and R is the distance to the rotation axis. Apparently, we have

$$FR_2 = (I_1 + I_2)\alpha\,, \tag{8.16}$$

which looks a lot like $F = (m_1 + m_2)a$ for common linear motion!

8.1.2 Rotational Kinetic Energy

Let's introduce another angular concept before we dive into their consequences in the following lectures. Let's again consider the two masses connected by a rigid rod:



In this system, we will have the center-of-mass travel with constant velocity $\vec{v}_{\rm cm} = v\hat{\jmath}$, and the masses orbit the center-of-mass at angular frequency ω. Let's calculate the kinetic energy of this system. The kinetic energy is simply

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\,, \tag{8.17}$$

but we need to find $\vec{v}_1$ and $\vec{v}_2$ given the data of the problem.

Let's first consider mass 1. We can break its velocity into the linear component and rotational component. The linear component is simply the center-of-mass velocity:

$$\vec{v}_{1,\rm lin} = v\hat{\jmath}\,. \tag{8.18}$$

For the rotational component, we had studied this long ago and can express the tangential velocity as

$$\vec{v}_{1,\rm rot} = -\omega R_1 \sin\omega t\,\hat{\imath} + \omega R_1 \cos\omega t\,\hat{\jmath}\,, \tag{8.19}$$

for example. Then, the total velocity of mass 1 is the sum of these expressions, where

$$\vec{v}_1 = (-\omega R_1 \sin\omega t)\,\hat{\imath} + (v + \omega R_1 \cos\omega t)\,\hat{\jmath}\,. \tag{8.20}$$

Its square is thus

$$\begin{aligned} v_1^2 = |\vec{v}_1|^2 = \vec{v}_1\cdot\vec{v}_1 &= \omega^2 R_1^2 \sin^2\omega t + v^2 + \omega^2 R_1^2 \cos^2\omega t + 2v\omega R_1 \cos\omega t \\ &= v^2 + \omega^2 R_1^2 + 2v\omega R_1 \cos\omega t\,. \end{aligned} \tag{8.21}$$

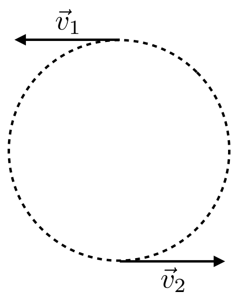
Then, the kinetic energy of mass 1 is

$$K_1 = \frac{1}{2}m_1 v^2 + \frac{1}{2}m_1 R_1^2\omega^2 + m_1 v\omega R_1 \cos\omega t\,. \tag{8.22}$$

Let's now do the same thing for mass 2. Mass 2's linear velocity is still the center-of-mass velocity,

$$\vec{v}_{2,\text{lin}} = v\hat{\jmath}\,. \tag{8.23}$$

For the rotational component of mass 2's velocity, note that its direction must be opposite to that of mass 1:



and the magnitude of $\vec{v}_2$ is determined by the angular velocity ω and its distance from the center-of-mass,

$$\vec{v}_{2,\text{rot}} = \omega R_2 \sin\omega t\,\hat{\imath} - \omega R_2 \cos\omega t\,\hat{\jmath}\,. \tag{8.24}$$

Then, the total velocity of mass 2 is

$$\vec{v}_2 = (\omega R_2 \sin\omega t)\,\hat{\imath} + (v - \omega R_2 \cos\omega t)\,\hat{\jmath}\,, \tag{8.25}$$

and its square is

$$\begin{aligned} v_2^2 &= \omega^2 R_2^2 \sin^2\omega t + v^2 + \omega^2 R_2^2 \cos^2\omega t - 2v\omega R_2 \cos\omega t \\ &= v^2 + \omega^2 R_2^2 - 2v\omega R_2 \cos\omega t\,. \end{aligned} \tag{8.26}$$

The kinetic energy of mass 2 is then

$$K_2 = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_2 R_2^2\omega^2 - m_2 v\omega R_2 \cos\omega t\,. \tag{8.27}$$

The total kinetic energy of the system is the sum of the kinetic energies of the two masses, where

$$K = K_1 + K_2 = \frac{1}{2}(m_1+m_2)v^2 + \frac{1}{2}m_1 R_1^2\omega^2 + \frac{1}{2}m_2 R_2^2\omega^2 + (m_1 R_1 - m_2 R_2)v\omega\cos\omega t\,. \tag{8.28}$$

Note that $m_1R_1 = m_2R_2$ because

$$m_1 \frac{m_2 d}{m_1+m_2} = m_2 \frac{m_1 d}{m_1+m_2}\,, \tag{8.29}$$

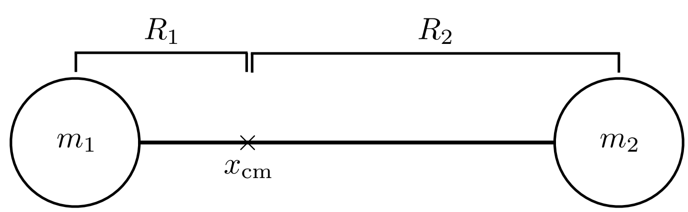
so the weird final term, with the naked cosine factor, vanishes! We can therefore write the kinetic energy of this system as

$$K = \frac{1}{2}(m_1+m_2)v^2 + \frac{1}{2}(I_1+I_2)\omega^2\,, \tag{8.30}$$

where $I_1 = m_1R_1^2$ and $I_2 = m_2R_2^2$, the moments of inertia of the two masses.

8.2 Calculating the Moment of Inertia

Previously, we studied the dumbbell system of two masses m_1 and m_2 connected together by a massless, rigid rod:

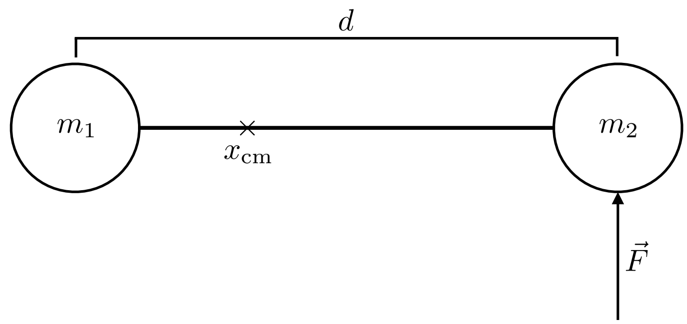


The center-of-mass of the system is identified and the distances of each mass to the center-of-mass is

$$R_1 = \frac{m_2 d}{m_1+m_2}\,, \qquad R_2 = \frac{m_1 d}{m_1+m_2}\,, \tag{8.31}$$

where the separation distance of the masses is d. We had considered exerting a force upward on mass 2, which had one consequence of accelerating the center-of-mass:

$$\vec{F} = (m_1+m_2)\vec{x}_{\rm cm}\,, \tag{8.32}$$



but also had the consequence of rotating the two masses about the center-of-mass. By considering the net force on mass 2 exclusively, we demonstrated that its Newton's second law could be written as

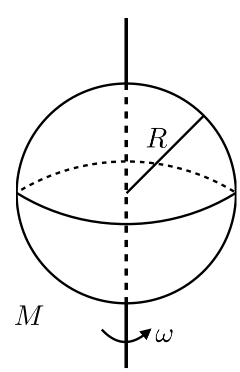
$$FR_2 = (m_1 R_1^2 + m_2 R_2^2)\alpha \,, \tag{8.33}$$

where α is the angular acceleration of the two masses about the center-of-mass and the quantity $mR^2 = I$ is called the moment of inertia.

The moment of inertia is the inertia of a mass m that impedes rotational change (i.e., angular acceleration) about an axis a distance R from the mass. For a point mass, the moment of inertia is just mR^2, and for an extended object, the moment of inertia can be found by simply summing over many small masses. In this lecture, we will explicitly calculate the moment of inertia of a sphere, about an axis that passes through its center. As we need to sum over a lot of masses and a sphere is a three dimensional object, we will need to do many integrals. Have no fear; we will break it down into many small steps.

8.2.1 Moment of Inertia of a Sphere

Let's first draw the sphere and rotation axis:



Let's give the sphere a radius R and total mass M, and the sphere is being rotated about the vertical axis through its center. For a small part of the sphere of mass dm, its moment of inertia dI is

$$dI = r^2 dm\,. \tag{8.34}$$

So our goal is to find r and dm and do the necessary integrals.

First, as we had done with the center-of-mass, let's break apart the small mass dm into a product of a mass density ρ and a small volume dV:

$$dm = \rho\, dV\,, \tag{8.35}$$

where

$$\rho = \frac{M}{\text{Vol}}\,, \tag{8.36}$$

where Vol is the total volume of the sphere. For a sphere of radius R, its volume is

$$\text{Vol} = \frac{4}{3}\pi R^3\,, \tag{8.37}$$

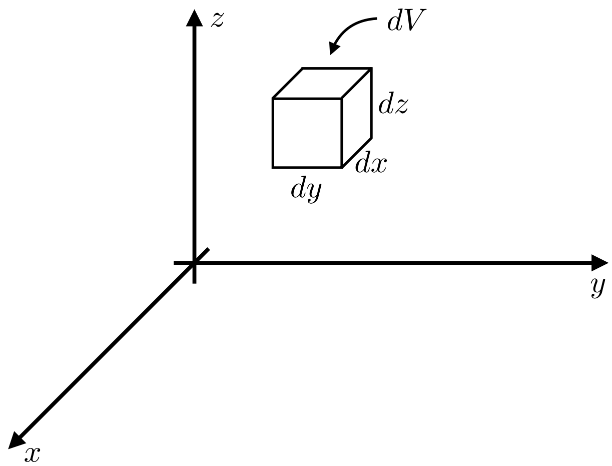
so the density is

$$\rho = \frac{3}{4}\frac{M}{\pi R^3}\,. \tag{8.38}$$

In what follows, we will just leave the density implicit as ρ, only plugging in the explicit expression at the end.

Now, our small moment of inertia is $dI = r^2\,\rho\, dV$, so we need to figure out the small volume dV, for some component of the sphere. To do this, we need a coordinate system, just like we need coordinates to express a position vector. As always in this business, life cannot imitate art, so the value of the volume is independent of your coordinates, but you need to represent it somehow to make progress.

One possible coordinate system is Cartesian coordinates in which we represent points by their x (left-right), y (forward-back), and z (up-down) position. This is likely the coordinate system you are most familiar with, as we have often employed it in these lectures. We can draw these coordinates and a small volume in this space as



A small volume in Cartesian coordinates is a parallelepiped of sides dx, dy, dz, so the volume is

$$dV = dx\, dy\, dz\,. \tag{8.39}$$

So, to calculate the volume of the sphere for example, we just sum up a bunch of little parallelepipeds, with the constraint that they form a sphere. The surface of a sphere is defined by the equation

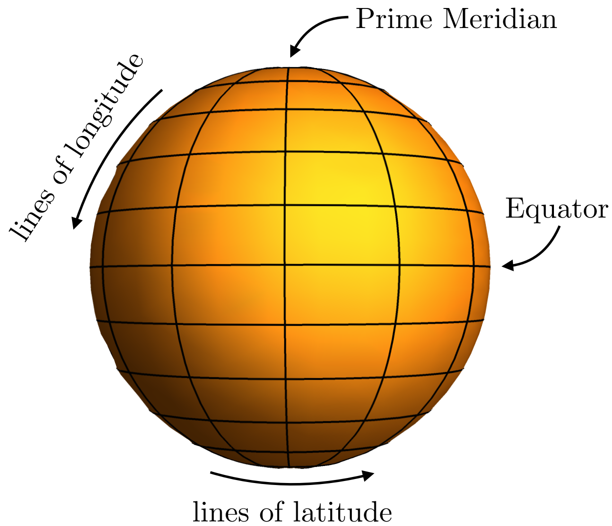
$$x^2 + y^2 + z^2 = R^2\,, \tag{8.40}$$

where R is the radius, and so points closer to the origin than the surface (the "bulk" of the sphere) are defined by the inequality

$$x^2 + y^2 + z^2 \leq R^2\,. \tag{8.41}$$

8.2.2 Spherical Coordinates

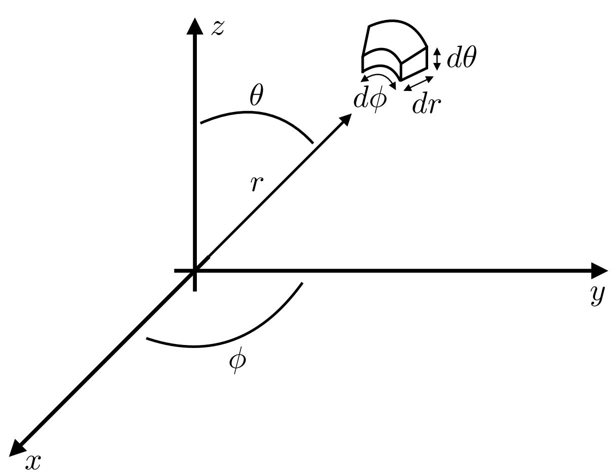
Again, I emphasize that life in physics cannot imitate art, so we could continue along this path and evaluate the moment of inertia using Cartesian coordinates. However, in practice, enforcing the relationship above is very inconvenient because a sphere is not well-approximated by a parallelepiped. So, instead of Cartesian coordinates, let's use coordinates to express the small volume in the sphere in a way most natural for the sphere. These coordinates, or at least two of them, are familiar to you from the expression of a location on the surface of the Earth. Rather than x, y, z coodinates augmented with the constraint of being on the surface of Earth, we use latitude and longitude to express a location. That is, we put a grid on the surface of Earth in terms of relative angle from the Equator and the Prime Meridian:



Given the value of the latitude θ and longitude ϕ, we can identify a unique point on the surface of Earth. Latitude is also called the **polar angle**, because it ranges between the poles, while longitude is an **azimuthal angle** as it varies about the axis defined by the poles.

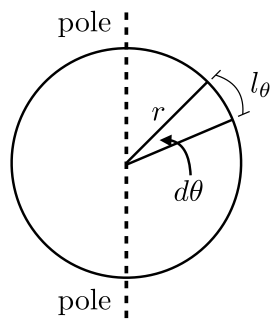
Further, for our sphere of interest that we want to calculate its moment of inertia, we need a radial coordinate r that varies from 0 (center of Earth) to R (surface of Earth). With r, θ, ϕ specified, we identify a unique point in the sphere. Also, note that restricting to the surface of Earth is very simple: we just require that $r = R$, with no squares or square-roots like in Cartesian coordinates.

With these coordinates identified, let's now figure out what the expression for the small volume dV is. Let's draw a picture:



The volume of this little chunk is then simply the product of the length of its three sides. Note that this is not simply $dr\,d\theta\,d\phi$, because, among other issues, volume has units of length-cubed, while this expression only has dimensions of length (angles are dimensionless). So we need to work a bit harder.

The length in the radial dimension of this chunk is indeed just dr, and to find the lengths in the θ and ϕ dimensions, we will consider projections of the sphere in different planes. Let's take a slice of the sphere along a line of longitude to determine the length in the θ dimension, l_θ:

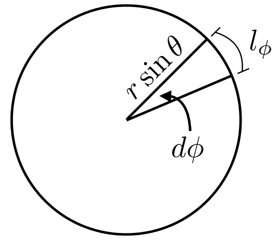


The arc length of an angular region of size $d\theta$ at radius r is simply

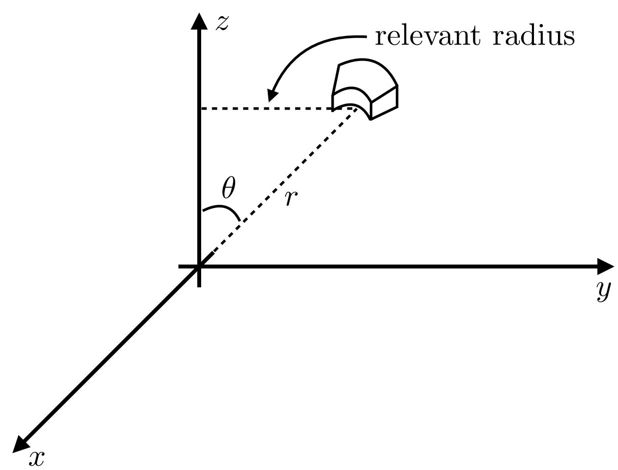
$$l_\theta = r\,d\theta\,, \tag{8.42}$$

which is what we need.

Now, let's take a slice along a line of latitude, an angle θ from the North Pole. The picture of this for determining the length in the ϕ dimension, l_ϕ, is



Note that now the radius of this slice is $r\sin\theta$ which you can see from the geometry of where the chunk is located:



It then follows that this arc length is

$$l_\phi = r \sin\theta \, d\phi \,. \tag{8.43}$$

Putting it all together, we find the volume of a small chunk in spherical coordinates to be

$$dV = dr \, dl_\theta \, dl_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi \,, \tag{8.44}$$

located a distance r from the origin and an angle θ from the North Pole. Whew!

8.2.3 Three-Dimensional Integration

Okay, now we just need to multiply this volume by the density ρ, then by the distance from the axis of rotation (the z axis) and integrate, and we have the moment of inertia. We had just identified the distance from the z axis in calculating the volume element of the chunk:

$$r_z = r \sin\theta \,, \tag{8.45}$$

so the moment of inertia of the chunk is

$$dI = \rho \, r_z^2 \, dV = \rho \, r^4 \, \sin^3\theta \, dr \, d\theta \, d\phi \,. \tag{8.46}$$

The total moment of inertia of the sphere is

$$I = \int dI = \rho \int_0^R r^4 \, dr \cdot \int_0^\pi \sin^3\theta \, d\theta \cdot \int_0^{2\pi} d\phi \,. \tag{8.47}$$

Note the simple product of one-dimensional integrals here. That will make life very simple (and something that would not have happened with Cartesian coordinates). Note also the bounds of integration: r ranges from 0 to R, the radius of the sphere; θ ranges from 0 radians (North Pole) to π radians (South Pole); and ϕ ranges from 0 to 2π radians (all the way around a circle). So, two of these integrals are

$$\int_0^R r^4\, dr = \frac{R^5}{5}\,, \qquad \int_0^{2\pi} d\phi = 2\pi\,. \tag{8.48}$$

The integral over θ can be evaluated using a u-substitution which I encourage you to derive on your own. The answer is

$$\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}\,. \tag{8.49}$$

With the density of

$$\rho = \frac{3}{4}\frac{M}{\pi R^3}\,, \tag{8.50}$$

we find the moment of inertia of the sphere to be

$$I_{\text{sphere}} = \frac{3}{4}\frac{M}{\pi R^3}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \frac{2}{5}MR^2\,. \tag{8.51}$$

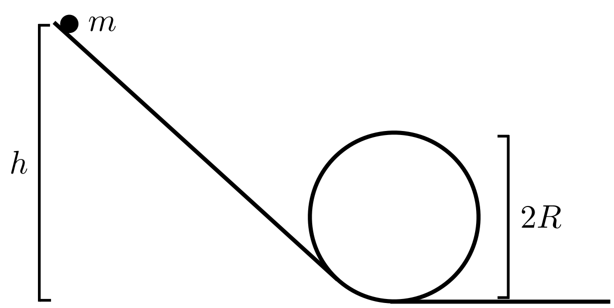
Whew!

8.3 Ball Through a Loop-the-Loop Redux

In the previous lectures, we motivated the moment of inertia as a measure of rotational inertia, analogous to mass as linear inertia that opposes changes to its motion. From one perspective, moment of inertia is just a short-hand for a complete analysis of the masses and velocities of objects undergoing rotational motion. One could just use standard Newton's laws to describe rotational motion, and everything would work out. However, we will see starting now that re-expressing Newton's laws, kinetic energies, forces, etc., in a rotational language will be extremely powerful and convenient.

8.3.1 Review

In this lecture, we will revisit a problem we had analyzed with energy conservation a while ago, now accounting for rotational motion as well. We are going back to the old loop-the-loop problem: given a ramp that leads to a loop-the-loop of radius R, what is the minimum height h that a ball should be released from to make it all the way around the ramp?



Let's remind ourselves of what we had done previously. First, assuming that the ball slides down the ramp without friction, we used conservation of energy to relate the initial gravitational potential energy to the energy at the top of the loop. We have:

$$mgh = \frac{1}{2}mv^2 + mg(2R)\,, \tag{8.52}$$

where the mass of the ball is m and the speed of the ball at the top of the loop is v. Now, for the ball to stay in the loop, it must be traveling in a circle, which enforces a minimum centripetal acceleration. At the top of the loop, centripetal acceleration is minimized if the ball just comes off the loop there, so the only force acting on it is gravity. So, demanding that the centripetal acceleration at the top of the loop is at least g, we find that the speed v is

$$g = \frac{v^2}{R}\,, \qquad \text{or that} \qquad v^2 = Rg\,. \tag{8.53}$$

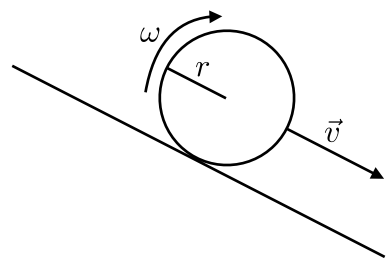
Inserting this into the conservation of energy from earlier, we then find

$$mgh = \frac{1}{2}m(Rg) + mg(2R)\,, \qquad \text{or that} \qquad h = \frac{5}{2}R\,. \tag{8.54}$$

We can test this, as we have done, and we find that the ball doesn't make it all the way around if we release it from a height of $h = \frac{5}{2}R$. We have to release the ball higher!

8.3.2 Including Rotational Kinetic Energy

So what is going on? Now, with our introduction to rotational dynamics, we know that there is more energy in the ball than just that from translation of it center-of-mass. As the ball goes down the ramp, it is both translating and rotating about its center:



Indeed, the ball is rolling without slipping, which interestingly implies that static friction is very important. The ball we roll down the ramp is a solid metal sphere (a ball bearing) so its total kinetic energy when rolling can be expressed as motion of its center-of-mass and rotation about its center-of-mass:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}mr^2\omega^2\,. \tag{8.55}$$

In the second equation, we inserted the expression for the moment of inertia of a sphere with mass m and radius r. Further, if the ball is rolling without slipping, we have a simple relationship between the speed and angular speed:

$$v = \omega r\,. \tag{8.56}$$

Then, the kinetic energy of the rolling ball is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}mr^2\omega^2 = \left(\frac{1}{2} + \frac{1}{5}\right)mv^2 = \frac{7}{10}mv^2\,. \tag{8.57}$$

The coefficient 7/10 is kind of weird, but if it works to describe our data/demonstration, then it isn't that weird.

So, with this result, our conservation of energy expression becomes

$$mgh = \frac{7}{10}mv^2 + mg(2R)\,. \tag{8.58}$$

We still have the requirement that the minimum centripetal acceleration is g so that

$$v^2 = Rg\,. \tag{8.59}$$

This then implies that

$$mgh = \frac{7}{10}mRg + mg(2R)\,, \tag{8.60}$$

or that the minimum height h that the ball must be released from to go around the loop-the-loop is

$$h = \frac{27}{10}R > \frac{5}{2}R\,. \tag{8.61}$$

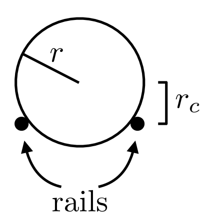
Let's test this out! (See `https://youtu.be/P_bMYJsrewo`)

8.3.3 Other Effects

Apparently, we need to go a little higher than even $27/10R$ to get the ball to go completely around the loop. Of course, we can wave our hands and say words like "sounds energy losses," "friction," and the like, but there are a couple of physics points that we have ignored that I want to ask about now.

Example

First, the ball isn't sitting on top of the ramp; it rides between two rails like so



Note that the distance from the rails to the axis of rotation of the ball, r_c, is less than the radius of the ball, r. How does this affect the minimum height from which the ball needs to be released?

(a) Increases h beyond $\frac{27}{10}R$

(b) Decreases h

(c) No effect on the minimum height

Okay, let's figure this out. Again, the kinetic energy of the rolling ball is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\omega^2\,. \tag{8.62}$$

Note, however, that the speed is not $v = r\omega$, but set by the smaller radius r_c:

$$v = r_c\omega\,. \tag{8.63}$$

Then, the kinetic energy of the ball is

$$K = \frac{1}{2}mv^2 + \frac{1}{5}mr\frac{r^2}{r_c^2}r_c^2\omega^2 = \left(\frac{1}{2}\frac{r^2}{r_c^2}\right)mv^2\,. \tag{8.64}$$

Our conservation of energy equation is therefore

$$mgh = \left(\frac{1}{2}\frac{r^2}{r_c^2}\right)mv^2 + mg(2R) = \left(\frac{1}{2}\frac{r^2}{r_c^2}\right)mgR + mg(2R)\,, \tag{8.65}$$

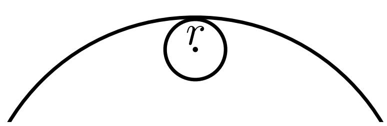
where we have inserted $v^2 = gR$ by the centripetal acceleration. Then, the minimum height h is

$$h = \left(\frac{5}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right)R > \frac{27}{10}R\,, \tag{8.66}$$

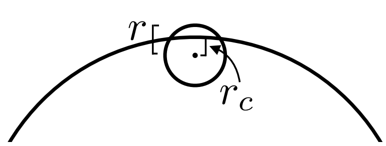
because $r > r_c$. So, we have to release the ball even higher.

Example

There is another effect we have ignored. Because of the finite size of the ball, its center-of-mass actually doesn't get to a height of $2R$ at the top of the loop. If the ball rode on top of the ramp, it would only be at a height of $2R - r$:



Accounting for the separated rails that the ball rides between, this height is increased to $2R - r_c$:



How does this displacement of the ball's center-of-mass affect the minimum height further?

(a) Increases h

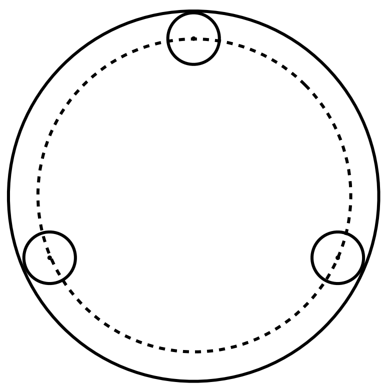
(b) Decreases h

(c) No effect on the minimum height

Because the center-of-mass only gets to a height of $2R - r_c$, this affects the conservation of energy expression:

$$mgh = \left(\frac{1}{2}\frac{r^2}{r_c^2}\right) mv^2 + mg(2R - r_c)\,. \tag{8.67}$$

Additionally, the radius of the circle through which the center-of-mass travels in the loop-the-loop is $R - r_c$, and not just R:



Then, the restriction on centripetal acceleration is

$$g = \frac{v^2}{R - r_c}\,, \qquad \text{or that} \qquad v^2 = (R - r_c)g\,. \tag{8.68}$$

Then, conservation of energy becomes

$$mgh = \left(\frac{1}{2}\frac{r^2}{r_c^2}\right) mg(R - r_c) + mg(2R - r_c)\,, \tag{8.69}$$

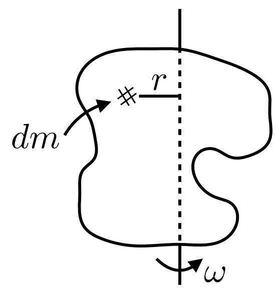
which slightly decreases the minimum value of h by a distance Δh:

$$\Delta h = -\left(\frac{3}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) r_c\,, \tag{8.70}$$

which is quite small compared to the increase in h from accounting for the rails.

8.4 The Right-Hand Rule

In the previous few lectures, we have introduced a formalism for describing the dynamics of a rotating rigid body. I want to emphasize that Newton's second law as we introduced it near the beginning of this course, as well as our expressions for kinetic energies, momenta, etc., can all be used to describe a rigid body that is rotating, but it is very convenient to rephrase expressions exclusively in terms of properties of rotation. For example, if you have an object that is rotating about the axis like so:

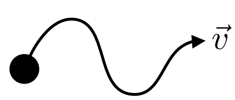


you can calculate its kinetic energy by summing up the kinetic energy of every little mass that composes it. This would be

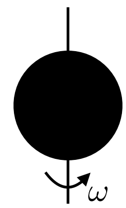
$$K = \frac{1}{2}\int v^2\, dm = \frac{1}{2}\int \omega^2 r^2\, dm = \frac{1}{2}\left(\int r^2\, dm\right)\omega^2 = \frac{1}{2}I\omega^2\,. \tag{8.71}$$

In the second equality, we note that a rigid body must rotate with the same angular velocity everywhere and r is the distance from little mass dm to the axis of rotation. Because ω is constant over the object, it can pull out of the integral and we identify that the integral that remains is just the moment of inertia of the object. So, it becomes much more convenient to express the kinetic energy of a rotating object as $\frac{1}{2}I\omega^2$, because we only have to calculate the moment of inertia once and for all.

Nevertheless, there was still something slightly odd about how we denoted the angular velocity of the object. We say "angular velocity," but I have never expressed it as a vector; that is, I have only written ω, not $\vec{\omega}$. This might seem a bit odd because "velocity" is definitely a vector. Further, how we have expressed the direction of rotation is a bit strange from the perspective of linear motion, for example. For an object traveling along a non-straight path, we wouldn't draw its relative velocity as



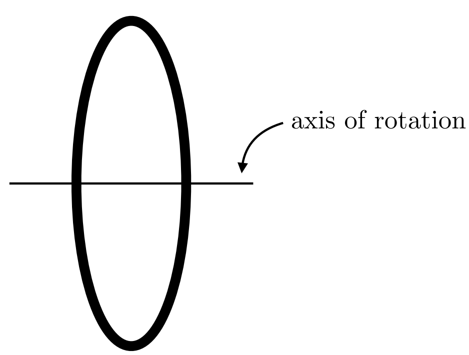
yet we seem comfortable drawing the direction of rotation of an object in a non-straight manner:



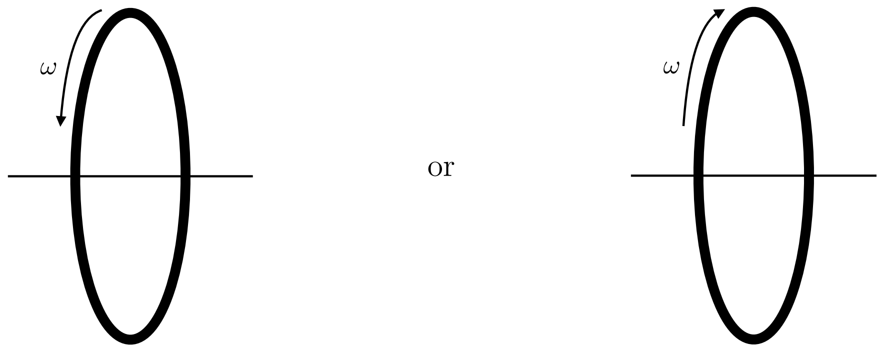
Yet another issue is that an object rotating at a constant rate should continue to rotate at a constant rate if there are no external forces acting on it. We've seen hints of this before, but the argument is simple: to rotate requires a force that keeps every point in the object in centripetal acceleration. This centripetal acceleration can be completely accounted for by internal forces in the system object; c.f., two mutually-orbiting bodies interacting gravitationally. As such, there should be no ambiguity for what the angular velocity is. We shouldn't need a wonky curvy arrow to denote it. So let's figure out a better notation.

8.4.1 Right-Hand Rule #1: Angular Velocity

For concreteness, let's consider rotating a bicycle tire about the axis through its center:

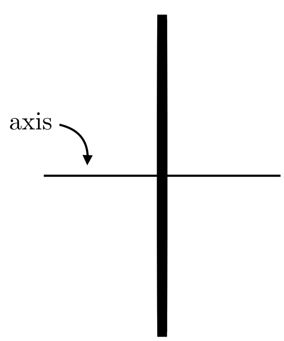


Given this axis of rotation, how many options are there for the direction of rotation? This is amazingly simple and profound: given an axis of rotation, there are only two options for the direction of rotation. The wheel can rotate with the front of the tire moving down or up from your perspective:

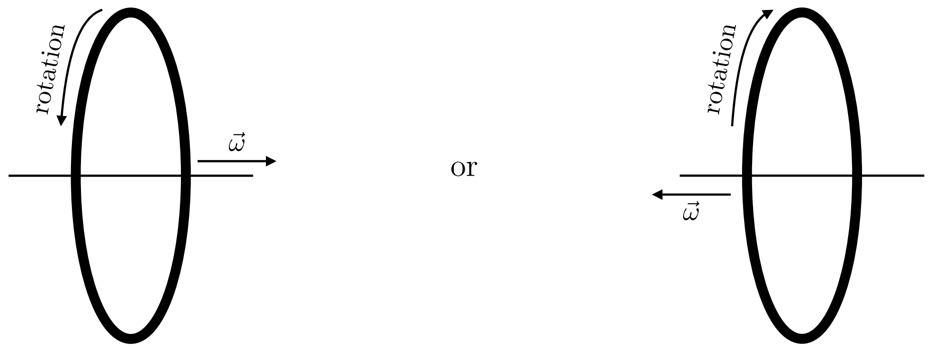


However, "moving up" or "moving down" is not a unique way to define rotation. If the wheel looks to be moving up to you, what does it look like to someone on the other side of the wheel? They see it moving down! So, we need another way to denote the direction of rotation.

Let's look at the wheel head-on:



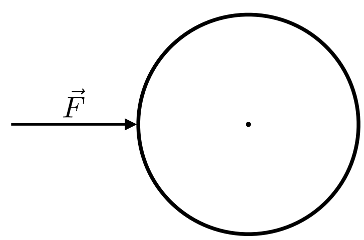
Note that there are the same numbers of directions of rotation (2) as sides of the wheel (2). Another way to say this is that given the axis of rotation, we can move along it to the left or to the right. So, a natural way to express the angular velocity is as a vector that points either to the left or the right along the axis. Then, the axis is clear and the direction of rotation is unambiguous, given a convention for mapping direction of rotation to a direction along the axis. This mapping is called the **right-hand rule**. What one does is curl the finger on your right hand in the direction of rotation and then your thumb points in the direction of the angular velocity vector, $\vec{\omega}$. For the wheel, we have



This is but one of the many right-hand rules in physics.

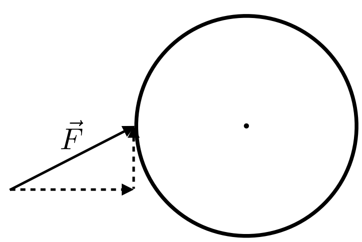
8.4.2 Right-Hand Rule #2: Torque

The second right-hand rule we will introduce now. How do we get the wheel rotating in the first place? Just like with getting an object to move from rest, we have to apply a force. Unlike for linear motion, however, just applying any old force won't do. For example, if I pushed the wheel perpendicular to it (radially), in the direction of the center of the wheel, would the wheel rotate? I only care if it rotates, not if it moves otherwise. That is, I apply a force

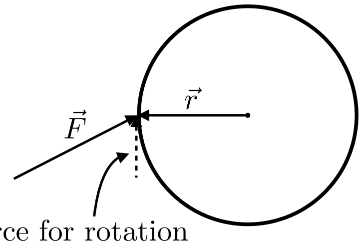


No! To convince yourself of this, try closing a door by pushing on the edge of the door, toward the hinge.

To get the wheel rotating, we have to apply a force with some non-zero component tangent to the surface of the wheel:



The component of the force in the direction of the axis of rotation is doing nothing to help actually rotate the wheel. So, the component of the force that works to rotate the wheel is perpendicular to the position vector at which the force is applied:



Further, more change in rotation is accomplished by pushing farther from the axis of rotation. If we push a door right at the hinge, it is very hard to close the door, which is why door handles are located at the outside edge of the door. Similarly, for the same force, there is much less change in rotation of the wheel if the force is applied near the axis of rotation. By the way, does that imply that higher or lower gears on a bike correspond to a force applied farther from the axis/axle?

So, if we only want the component perpendicular to the position with respect to the axis and we want to be further away from the axis to rotate more easily, this suggests that the "rotational force" or **torque** τ has magnitude

$$\tau = Fr\sin\theta\,, \tag{8.72}$$

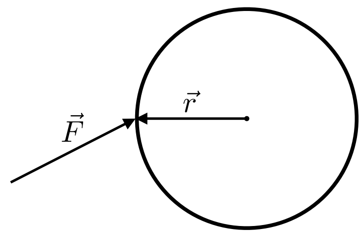
where θ is the angle between $\vec{F}$ and $\vec{r}$. This torque is the agent that enacts rotational change, so, by Newton's second law, is equal to the product of rotational inertia and angular acceleration:

$$\tau = I\alpha\,, \tag{8.73}$$

where I is the moment of inertia (rotational inertia) and α is the angular acceleration. This is referred to as Newton's second law for rotations. For forces acting on two masses connected by a rigid rod, we had derived this expression simply through manipulations of Newton's second law for linear motion.

Example

This is great, but it's not a vector equation yet. We need to figure out the direction of angular acceleration that a given torque induces. Let's consider a force on the tire as such



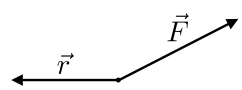
What direction will the wheel rotate? So therefore what is the direction of angular velocity?

(a) no rotation

(b) $\vec{\omega} \otimes$ (into the page)

(c) $\vec{\omega} \odot$ (out of the page)

Let's place the force and position vectors with their tails together:



This force will enact a rotation clockwise from our perspective, so angular velocity or acceleration are both into the page. We used the right-hand rule to find that direction of angular velocity; can we construct a right-hand rule with $\vec{r}$ and $\vec{F}$ that produces the same direction? Indeed, and the right-hand rule for the direction of torque is:

1. Place your left hand behind your back.
2. Point the fingers of your right hand in the direction of position vector $\vec{r}$.
3. Curl the fingers of your right hand in the direction of force vector $\vec{F}$.
4. The thumb of your right hand points in the direction of torque $\vec{\tau}$.

Does this work for our example here? Remember, it is called the *right-hand rule* for a reason: you have to use your right hand! (See step 1 above.) This isn't some conspiracy against southpaws, we just need some convention for defining the direction of angular vectors.

To end this discussion, I want to express Newton's second law for rotations in vector form. It should encode the direction information we have discussed and this can be accomplished by a **vector cross product**,

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}\,. \tag{8.74}$$

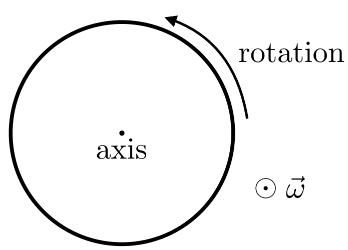
The magnitude of the cross product of two vectors is

$$|\vec{r} \times \vec{F}| = rF \sin\theta\,, \tag{8.75}$$

where θ is their relative angle.

8.5 Static Equilibrium

Previously, we had introduced the angular velocity and torque as honest vectors, identifying the direction associated with them. This was non-trivial and unfamiliar from our analysis of vectors like velocity and force, because angular velocity and torque describe properties of rotation in a plane. Our solution to identifying a unique direction for these vectors was to note that they should encode information about the axis of rotation, and the direction of rotation about the axis of rotation. Given an axis of rotation, an object can rotate one of two directions, and so we identify the angular velocity vector as pointing in one of two directions down the axis of rotation. In practice we find the direction of $\vec{\omega}$ using the right-hand rule: point the fingers on your right hand in the direction of rotation, curl them in the direction of rotation about the axis, and your thumb points in the direction of $\vec{\omega}$:

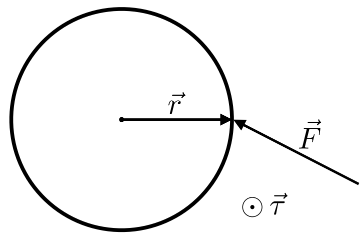


We identified torque as the application of a force on an extended object that acts to change the rotational motion of that object. As such, there is a corresponding Newton's law for rotation

$$\vec{\tau} = I\vec{\alpha}\,, \tag{8.76}$$

where $\vec{\tau}$ is the torque, I is the object's moment of inertia, and $\vec{\alpha}$ is the angular acceleration.

Torque is not just a force; it depends on how and where on an object it is applied, with respect to the axis of rotation. For example, a bicycle tire where we apply a force as so:



Only the component perpendicular to the radial vector $\vec{r}$ of the force $\vec{F}$ acts to rotate the tire. This component can be extracted with the vector cross product,

$$\vec{\tau} = \vec{r} \times \vec{F}\,, \tag{8.77}$$

which has magnitude $\tau = rF\sin\theta$, where θ is the angle between $\vec{r}$ and $\vec{F}$. The direction of torque is also found from a right-hand rule. This time, the rule is to point the fingers on your right hand in the direction of $\vec{r}$, curl in the direction of $\vec{F}$, and your thumb points in the direction of torque or angular acceleration.

8.5.1 Saqsaywaman

Now, with that long set-up and review, let's discuss some consequences. To motivate this lecture's topic, I want to take a brief digression and discuss the Saqsaywaman site near Cuzco, Peru. Saqsaywaman is a millenium-old city located in the mountains above Cuzco, of which all that remains are enormous stones terraced on the hillside and placed with exquisite precision on one another. An overview photograph of Saqsaywaman that I took is shown in Fig. 8.1, while a photograph of the detail of the stone placement is shown in Fig. 8.2.

So how does one go about building a structure like Saqsaywaman? We don't know actually how it was done almost a thousand years ago, but we can answer the question of how we would build it now, knowing the laws of physics formalized as we have discussed. If we want to construct a wall to stand for a thousand years, then an obvious requirement is that we want the wall, and every element of the wall, to be at rest. Again, "at rest" means that the center-of-mass of an object is not moving. If the center-of-mass is not moving, then necessarily the net force on that object is 0:

$$\vec{F}_{\text{net}} = 0\,. \tag{8.78}$$

For a wall, we need the net force of *every* element of the wall to be 0.

Figure 8.1: Photograph of the Saqsaywaman site above Cusco, Peru.

It's not enough to just have the net force on an object vanish for the wall to stand. A wall is a very boring object: it looks identical at any time after it is constructed. We know of systems for which their center-of-mass is at rest, and yet change in time. For example, we had studied two massive objects mutually orbiting their common center-of-mass through gravitation. There were no external forces, so the center-of-mass was at rest, yet the rotation about the center-of-mass meant that the system had some non-trivial time dependence. If you come back at some later time, you would see the masses in a different position than originally. So, if a wall is not to move in any way, we must also require that it has no motion/rotation about the center-of-mass. If this is the case, then the net torque on the object is 0:

$$\vec{\tau}_{\rm net} = 0\,. \tag{8.79}$$

Again, for a wall, we need the net torque of every element of the wall to be 0.

So, compactly, we require that every element of the wall satisfies:

$$\vec{F}_{\rm net} = 0\,, \qquad \Rightarrow \qquad \text{no motion of the center-of-mass,}$$

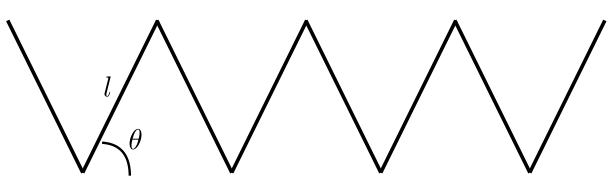
Figure 8.2: Detail of the stone placement at Saqsaywaman.

$$\vec{\tau}_{\text{net}} = 0 \,, \qquad \Rightarrow \qquad \text{no motion about the center-of-mass.}$$

These two conditions are sufficient to enforce no motion of the object whatsoever. When they are both true, an object is said to be in **static equilibrium**. To a large extent, the job of a civil engineer, who designs roads, bridges, buildings, and other infrastructure, is to design a structure that efficiently and acceptably remains in static equilibrium.

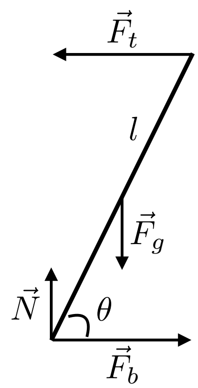
8.5.2 Detailed Study of a Simpler Example

As an example of a system in static equilibrium, let's analyze a structure a bit simpler than Saqsaywaman. We imagine the following: long, thin stones have been placed in a circle and lean at an angle to come to a point with the next stone in the circle. That is, a segment of the circle looks like



with the length of the stones l and the angle between the stones and the ground θ. The stones stay standing because they mutually exert forces on each other, similar to how a group of people can sit on each other's laps in a circle, with no one person holding up (or being held up by) more than one other person. For simplicity, we will assume no friction anywhere; can this ring stay standing?

Let's just focus on one stone and draw the forces exerted on it



Here, I have denoted gravity acting at the center of the stone, the normal force $\vec{N}$ from the ground, and additionally two normal forces, $\vec{F}_t$ and $\vec{F}_b$, exerted by the neighboring stones at the top and bottom of this stone, respectively. The ring of such stones can stay up if all forces acting on one stone are finite (or less than some structural maximum).

Let's first analyze $\vec{F}_{\text{net}} = 0$. Note that two pairs of forces act exclusively vertically ($\vec{N}$ and $\vec{F}_g$) and two exclusively horizontally ($\vec{F}_t$ and $\vec{F}_b$). Therefore, $\vec{F}_{\text{net}} = 0$,

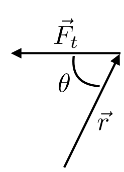
$$\vec{N} + \vec{F}_b + \vec{F}_t + \vec{F}_g = 0\,, \tag{8.80}$$

reduces to a statement about magnitudes

$$N = F_g = mg\,, F_t = F_b \equiv F\,. \tag{8.81}$$

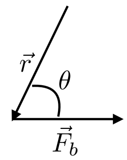
That was pretty easy; what about torques? We will analyze torques in two ways, about two different axes, but demonstrate that we get the exact same results, as we must. First let's consider torques about the center-of-mass axis. About this axis, $\vec{F}_g$ exerts no torque, while the torques of the other forces are

$\vec{F}_t$:



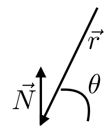
$\vec{\tau}\ \odot \qquad \tau = \frac{l}{2} F_t \sin\theta = \frac{l}{2} F \sin\theta$

$\vec{F}_b$:



$\vec{\tau}\ \odot \qquad \tau = \frac{l}{2}F_b \sin\theta = \frac{l}{2}F\sin\theta$

$\vec{N}$:



$\vec{\tau}\ \otimes \qquad \tau = \frac{l}{2}N\cos\theta = \frac{l}{2}mg\cos\theta$

Accounting for the direction of torques, demanding that the net torque be zero enforces

$$\sum \vec{\tau} = 0 \qquad \Rightarrow \qquad \frac{l}{2}F\sin\theta + \frac{l}{2}F\sin\theta = \frac{l}{2}mg\cos\theta\,, \tag{8.82}$$

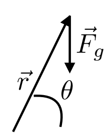
or that

$$F = \frac{mg}{2}\cot\theta\,. \tag{8.83}$$

That is, as $\theta \to 0$ (the stones get closer to the ground), the force they exert on each other increases. There will be some angle at which the force is so strong that the stones will break, so they should be placed nearly vertically to minimize this force.

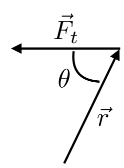
Finally, let's quickly analyze the not torque about another axis. Let's consider the axis at the bottom of the stone. Now, the normal force $\vec{N}$ and $\vec{F}_b$ have 0 torque as they are applied at the axis. The torques of the gravitational force and $\vec{F}_t$ are

$\vec{F}_g$:



$\vec{\tau}\ \otimes \qquad \tau = \frac{l}{2}mg\cos\theta$

$\vec{F}_t$:



$\vec{\tau}\ \odot \qquad \tau = lF\sin\theta$

so the net torque about the axis at the bottom of the stone enforces

$$\sum \vec{\tau} = 0 \qquad \Rightarrow \qquad lF\sin\theta = \frac{l}{2}mg\cos\theta\,, \tag{8.84}$$

or that

$$F = \frac{mg}{2}\cot\theta\,, \tag{8.85}$$

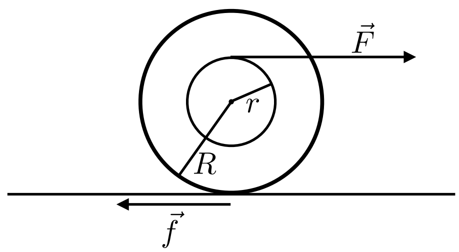
which is the same requirement we found earlier.

8.6 Rolling Without Slipping

With this understanding of rotational dynamics developed, we are going to have some fun interpreting the connection between rotational and translational motion.

8.6.1 Pulling a Spool Two Ways

To start, let's analyse pulling a rope connected to a spool along the ground with a force $\vec{F}$, without slipping. The picture of this is



The radius of the spool is R, while the radius of the inner section where the rope is wrapped around is $r < R$. I have denoted the pull force $\vec{F}$ and the force of friction $\vec{f}$, responsible for the rolling without slipping. Forces that act in the vertical direction (normal force and gravity) will be irrelevant to this discussion, so we ignore them. With this set-up, let's begin.

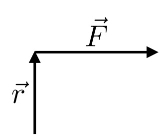
First, let's find the net force on the spool, so find the acceleration of the center-of-mass of the spool. We have

$$\vec{F} + \vec{f} = m\vec{a}\,, \tag{8.86}$$

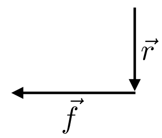
where m is the mass of the spool. In components, this is just

$$F - f = ma\,, \tag{8.87}$$

along the horizontal axis. Now, let's identify the torques on the spool. The torques from both $\vec{f}$ and $\vec{F}$ act in the same direction



$$\vec{r} \times \vec{F} = \vec{\tau}_F \quad \otimes$$



$$\vec{r} \times \vec{f} = \vec{\tau}_f \quad \otimes$$

by the right hand rule. Recall that the right hand rule is:

1. Place your left hand behind your back.
2. Point fingers on your right hand in the direction of $\vec{r}$, the vector that stretches from the axis of rotation to the point of application of the force.
3. Curl fingers in direction of applied force.
4. Thumb points in the direction of torque.

Additionally, the angle between the vectors $\vec{r}$ and $\vec{F}$ (or $\vec{f}$) are both $\theta = 90°$, so the magnitudes of the torques are

$$\tau_F = rF\,, \qquad\qquad \tau_f = Rf\,. \tag{8.88}$$

Then, Newton's second law for the rotation of the spool is

$$\tau_F + \tau_f = rF + Rf = I\alpha\,, \tag{8.89}$$

where I is the moment of inertia of the spool and α is its angular acceleration.

Now, if the spool rolls without slipping on the ground, this relates linear and angular accelerations a and α, where

$$a = R\alpha\,, \tag{8.90}$$

so that the torque is

$$rF + Rf = I\frac{a}{R}\,. \tag{8.91}$$

Now, we have two equations (net force and net torque) and two unknowns (a and f). Isolating f in the net force equation, we have

$$f = F - ma\,, \tag{8.92}$$

and plugging it into the net torque equation, we have

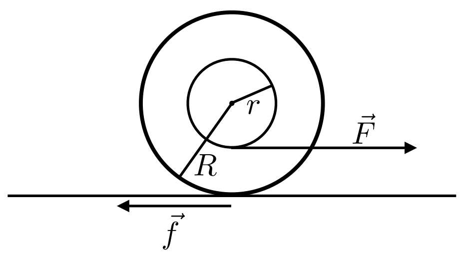
$$rF + R(F - ma) = \frac{I}{R}\,a\,, \tag{8.93}$$

or that the acceleration of the spool is

$$a = \frac{r+R}{mR + \frac{I}{R}}\,F\,, \tag{8.94}$$

to the right. Not surprisingly, if the mass or moment of inertia of the spool increases, this acceleration decreases, for the same force. We won't worry about solving for the moment of inertia here.

Now, I want to do something radical: what happens if I pull the spool as before, but now flip it over? That is, the setup now is:



Does the spool

(a) accelerate right,

(b) accelerate left, or

(c) does not accelerate.

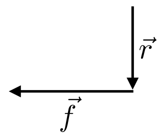
Well, let's analyze this system identically to what we did before. The net force equation is

$$\vec{F} + \vec{f} = m\vec{a}\,, \tag{8.95}$$

or in components,

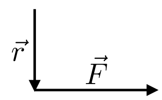
$$F - f = ma\,. \tag{8.96}$$

The torque of the friction is identical to earlier



$$\vec{r} \times \vec{f} = \vec{\tau}_f \quad \otimes$$

Now, the torque from the force $\vec{F}$ is in the opposite direction, by the right-hand rule



$$\vec{r} \times \vec{F} = \vec{\tau}_F \quad \odot$$

So now, Newton's second law for rotations is

$$\vec{\tau}_F + \vec{\tau}_f = I\vec{\alpha} \qquad \Rightarrow \qquad \tau_f - \tau_F = Rf - rF = I\alpha\,. \tag{8.97}$$

As before, we eliminate f from the net force equation and relate angular and linear accelerations with a radius factor,

$$R(F - ma) - rF = \frac{I}{R}\,a\,. \tag{8.98}$$

Rearranging and solving for a, we find

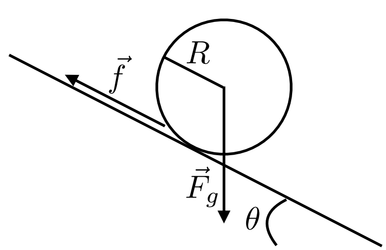
$$a = \frac{R - r}{mR + \frac{I}{R}}\,F\,. \tag{8.99}$$

Because $r < R$, this is still positive, so the spool still accelerates right. However, its magnitude is less than earlier because of the subtracted factor in the numerator.

Let's test this out! (See `https://youtu.be/Zax2Qew11dE`)

8.6.2 Rolling Down a Ramp

Finally for this chapter, I want to analyze the dynamics of an object rolling down a ramp:



Here, I have drawn the relevant forces for acceleration down the ramp (no normal force), and the angle of the ramp is θ, while the radius of the object is R. We will also assume that the mass and moments of inertia of the object are m and I, respectively.

As before, let's analyze the relevant forces that accelerate the object down the ramp:

$$\sum \vec{F}_x = m\vec{a}_x \qquad \Rightarrow \qquad F_g \sin\theta - f = mg\sin\theta - f = ma\,, \tag{8.100}$$

where a is the acceleration of the center-of-mass of the object.

Now, let's analyze torques for rotation about the center-of-mass of the object. Gravity has no torque because it acts at the center-of-mass; its lever arm is 0. Further, though not illustrated, normal force also exerts no torque because it acts at the surface of the object in the direction of its axis of rotation. The only force that exerts a torque is friction, so

$$\tau_f = fR = I\alpha = I\frac{a}{R}\,, \tag{8.101}$$

where we replaced angular acceleration α with linear acceleration via $a = R\alpha$, because the object does not slip.

Solving for friction f in the torque equation, we find

$$f = \frac{I}{R^2}\,a\,, \tag{8.102}$$

and inserting it into the force equation we have

$$mg\sin\theta - \frac{I}{R^2}a = ma\,, \tag{8.103}$$

or, solving for acceleration a,

$$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}\,. \tag{8.104}$$

Thus, we immediately see that objects that roll without slipping accelerate down a ramp more slowly if they have a larger moment of inertia, I. However, two objects, identical in shape, but that differ in size and mass, accelerate down the ramp identically.

Let's test this out! (See `https://youtu.be/SpJVbuJM9ks`)

Chapter 9

Angular Momentum

In this chapter, we will finish our discussion of rotational dynamics with a discussion of the final space-time conservation law.

9.1 Conservation of Angular Momentum

Let's start by reminding ourselves about the Newton's second law for rotational motion:

$$\vec{\tau}_{\text{net}} = I\vec{\alpha}\,, \tag{9.1}$$

where $\vec{\tau}_{\text{net}}$ is the net torque about a defined axis, I is the moment of inertia of the system about that axis, and $\vec{\alpha}$ is the angular acceleration. As an angular acceleration, $\vec{\alpha}$ is the time derivative of the angular velocity:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}\,. \tag{9.2}$$

9.1.1 Newton's Second Law with Angular Momentum

If the moment of inertia if constant, $\frac{dI}{dt} = 0$, we can re-write this Newton's second law as

$$\vec{\tau}_{\text{net}} = I\frac{d\vec{\omega}}{dt} = \frac{d}{dt}(I\vec{\omega})\,. \tag{9.3}$$

In this form, it is very similar to Newton's second law for linear motion:

$$\vec{F}_{\text{net}} = \frac{d}{dt}(m\vec{v})\,, \tag{9.4}$$

where we called $m\vec{v} \equiv \vec{p}$, the (linear) momentum. For Newton's second law for rotational motion, we instead refer to $I\vec{\omega}$ as the **angular momentum** $\vec{L}$,

$$\vec{L} \equiv I\vec{\omega}\,, \tag{9.5}$$

and so Newton's second law for rotations can be equivalently written as

$$\vec{\tau}_{\text{net}} = \frac{d}{dt}\,\vec{L}\,. \tag{9.6}$$

That is, changes in angular momentum are enacted by external torques (just like forces affect linear momentum).

Note that, by Newton's third law, the torques internal to a system always cancel pairwise, just like we observed with linear momentum. So, if there are no external torques, then the time derivative of angular momentum is 0:

$$\frac{d}{dt}\,\vec{L} = 0\,, \qquad \text{if} \qquad \vec{\tau}_{\text{net}} = \vec{\tau}_{\text{ext}} = 0\,. \tag{9.7}$$

That is, angular momentum is conserved if there are no external torques.

9.1.2 Spatial and Temporal Symmetries and Their Conservation Laws

Uh oh, we have a new conservation law, so you know that that means a digression into its consequences by Noether's theorem. Angular momentum conservation means that rotation about (at least) one axis is unchanged in time, defined by $\vec{L} = I\vec{\omega}$. If a system has non-zero angular momentum, then that system sweeps through arbitrary angles. Equivalently, the orientation of the system continually changes. If angular momentum is conserved, then there is no special angle or orientation; all orientations are equivalent and exhibit the same laws of physics. This is just to say that conservation of angular momentum means that all orientations are equivalent, or that there exists a symmetry of spatial rotation of our system.

So we have identified the following symmetries and conservation laws in our study this course:

Symmetry	Conservation Law
Time Translation	Energy
Spatial Translation	Momentum
Spatial Rotation	Angular Momentum

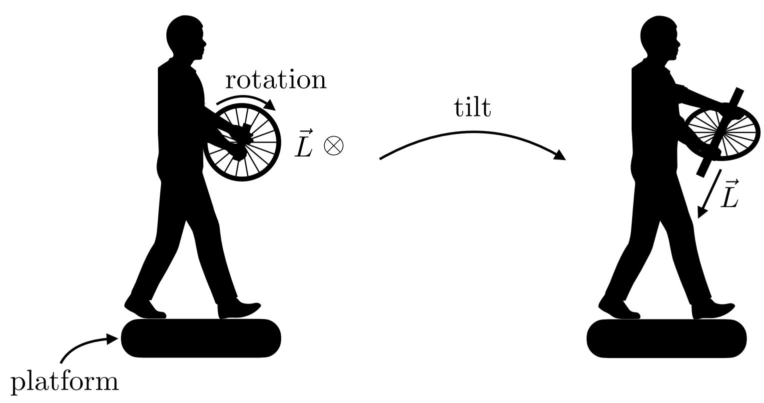
We believe that in our universe, energy, momentum, and angular momentum are all conserved. So, with that assumption, what does our universe look like if it is unchanged by the actions of time translation, spatial translation, and/or spatial rotation? I'll leave that for you to think about.

9.1.3 Rotation About Two Orthogonal Axes

Coming back from esoterica, let's see if we can understand some consequences of angular momentum conservation.

Example

As a concrete example, I'm going to perform the following demonstration. I am first going to get our old friend the bicycle wheel spinning pretty fast. Then, I will stand on a platform that is free to rotate and then I will pick up the wheel. Now, with the wheel in hand, I will tilt the wheel, so it makes a non-right angle with respect to the ground. A picture of this is



What happens?

(a) Nothing

(b) I start rotating with $\vec{\omega} \uparrow$

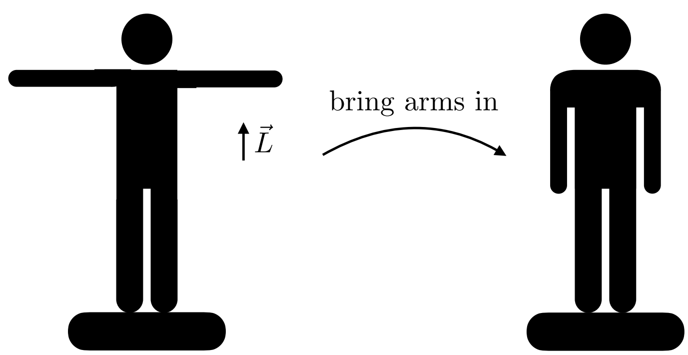
(c) I start rotating with $\vec{\omega} \downarrow$

Initially, the direction of angular momentum was into the page, $\otimes \vec{L}$. After I tilted the wheel, its angular momentum picked up a component in the downward vertical direction.

However, my tilting the wheel was an internal torque to the system of the wheel and me, and there were no relevant external torques. Therefore, angular momentum will be conserved. That is, the direction of the combined angular momentum of me and the wheel must still point into the page. To accomplish this, I must have an angular momentum that points upward, to cancel the change in angular momentum of the wheel.

Example

That's not the only way to affect angular momentum. Let's consider another demonstration; this time, where I am rotating standing on the platform with my arms outstretched. Then, I will bring my arms in close to my body. An illustration of this is



What happens when I do this?

(a) Angular velocity increases

(b) Angular velocity decreases

(c) Angular velocity stays the same

As with tilting the wheel, bringing my arms in is a completely internal action to the system of, well, me, so it does not affect my angular momentum. Therefore, my angular momentum before and after moving my arms is unchanged. However, with my arms outstretched, I have mass far from the axis of rotation and so by bringing them close to my body, I can decrease my moment of inertia significantly. So, if my initial moment of inertia is larger than my final moment of inertia, $I_i > I_f$, and angular momentum is unchanged,

$$\vec{L}_i = \vec{L}_f \qquad \Rightarrow \qquad I_i\vec{\omega}_i = I_f\vec{\omega}_f \,, \tag{9.8}$$

my angular velocity must increase for this equation to hold.

Let's try these out! (See `https://youtu.be/InM_mdY-6FI`)

9.1.4 Angular Momentum for Linear Motion

We will come back next section with more, crazier, predictions from Newton's second law for rotations. But for the rest of this section, I want to introduce another, equivalent, definition of angular momentum that significantly extends its realm of applicability.

Let's go back to the definition of torque:

$$\vec{\tau} = \vec{r} \times \vec{F}\,, \tag{9.9}$$

where $\vec{r}$ is the **lever arm**, the position vector that stretches from the axis of rotation to the point at which the force $\vec{F}$ is applied. Now, using linear Newton's second law, we can replace $\vec{F}$ with $\frac{d\vec{p}}{dt}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}\,. \tag{9.10}$$

If the lever arm vector $\vec{r}$ is constant in time, $d\vec{r}/dt = 0$, we can move the derivative all the way to the left:

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)\,. \tag{9.11}$$

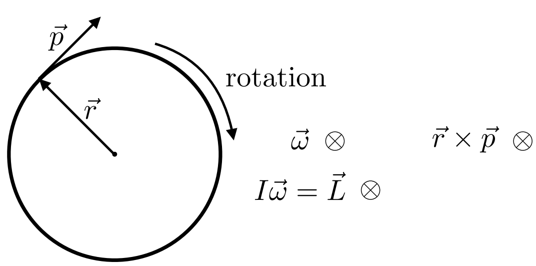
We had already established that torque is equal to the time derivative of angular momentum,

$$\vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)\,, \tag{9.12}$$

therefore, we have another definition of angular momentum, where

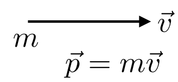
$$\vec{L} = \vec{r} \times \vec{p}\,. \tag{9.13}$$

Let's see if this makes sense for our rotating wheel:

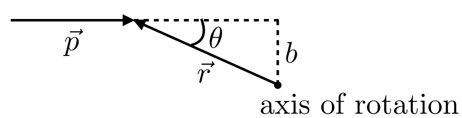


What about $\vec{r}\times\vec{p}$? Again, $\vec{r}$ is the vector from the axis of rotation to a point on the wheel that is rotating. The momentum of such a point on the wheel is in the direction tangent to the wheel. A point on the wheel is moving in a circle, and the direction of velocity/momentum of circular motion is tangent to the circle. So, to determine the direction of $\vec{r} \times \vec{p}$, we use the right-hand rule: point our fingers in the direction of $\vec{r}$, curl in the direction of $\vec{p}$, and our thumb points in the direction of $\vec{r} \times \vec{p} = \vec{L}$. This is exactly what we get with $\vec{L} = I\vec{\omega}$ and the right-hand rule for angular velocity!

This definition of angular momentum as $\vec{r} \times \vec{p}$ enables us to define angular momentum even for linear motion of a particle. Let's imagine that a particle is mass m with velocity $\vec{v}$ is passing by



There are no external forces on the particle, so it travels in a straight line. Now, we can just pick some arbitrary point in space and call it the "axis of rotation":



What is the angular momentum about this axis of rotation? First, the direction is

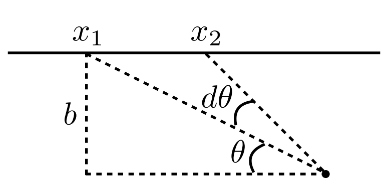
$$\vec{L} = \vec{r} \times \vec{p} \; \otimes \,, \tag{9.14}$$

into the page, by the right-hand rule. Next, its magnitude is

$$|\vec{L}| = |\vec{r} \times \vec{p}| = rp\sin\theta \equiv pb\,. \tag{9.15}$$

The perpendicular distance $b = r\sin\theta$ is called the **impact parameter**, and is the distance of closest approach of the particle to the axis of rotation. Because there are no external forces, the impact parameter is constant for a given trajectory, and so too is angular momentum constant as there are no net torques.

As to why linear motion can correspond to non-zero angular momentum, note that as the particle travels, it sweeps out angles with respect to the axis of rotation over time:



The distance $x_2 - x_1$ that the particle travels in time dt can be expressed as

$$x_2 - x_1 = \frac{b}{\tan(\theta + d\theta)} - \frac{b}{\tan\theta} \approx \frac{b}{\tan\theta}\left(\frac{1}{1 + \frac{d\theta}{\cos\theta\sin\theta}} - 1\right) \approx -\frac{b\,d\theta}{\sin^2\theta}\,, \tag{9.16}$$

with the approximation that the change in angle $d\theta$ is very small, $d\theta \ll \theta$. Note that it takes time

$$v\,dt = x_2 - x_1 = \frac{p}{m}\,dt \tag{9.17}$$

to travel this distance. Equating these expressions for the distance, we have

$$\frac{b\,d\theta}{\sin^2\theta} = \frac{p}{m}\,dt \qquad \Rightarrow \qquad \frac{mb^2}{\sin^2\theta}\frac{d\theta}{dt} = pb = L\,, \tag{9.18}$$

where we note that

$$\frac{d\theta}{dt} = \omega\,, \tag{9.19}$$

and the distance r from the axis of rotation to the particle's location is

$$r = \frac{b}{\sin\theta}\,, \tag{9.20}$$

so the moment of inertia is

$$I = mr^2 = m\,\frac{b^2}{\sin^2\theta}\,, \tag{9.21}$$

as expected!

9.2 Precession

Previously, we had rewritten Newton's second law for rotations in terms of the change imparted on angular momentum $\vec{L}$,

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}\,, \tag{9.22}$$

where $\vec{L} = I\vec{\omega} = \vec{r} \times \vec{p}$, the product of the moment of inertia and angular velocity or the vector cross product of the position vector $\vec{r}$ from the axis of rotation to the location of the particle which carries momentum $\vec{p}$. If there are no relevant external torques, $\vec{\tau}_{\text{net}} = 0$, angular momentum is conserved or unchanging in time. By Noether's theorem, angular momentum conservation means that the laws of physics are independent of orientation about the identified axis of rotation. In our universe, we believe that angular momentum is conserved so this orientation-independence has profound consequences on the structure of the universe.

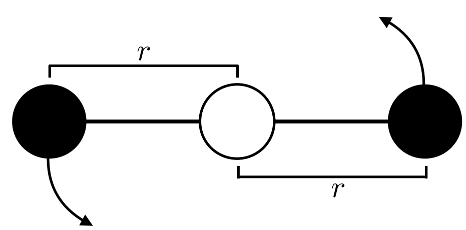
In this final lecture on rotations, we will relax the assumption of $\vec{\tau}_{\text{net}} = 0$ to understand the physics of a non-trivial Newton's second law for rotations.

9.2.1 Forces Applied Parallel to Angular Momentum

The first thing we will do is to understand a system in which $\vec{\tau}_{\text{net}}$ is secretly zero. Let's say I am rotating on the non-OSHA approved rotating platform with my arms outstretched, holding 2 kg weights in each hand. While rotating like this, I release the weights. What happens? Does my rotation rate

(a) Increase (b) Decrease (c) Stay the same

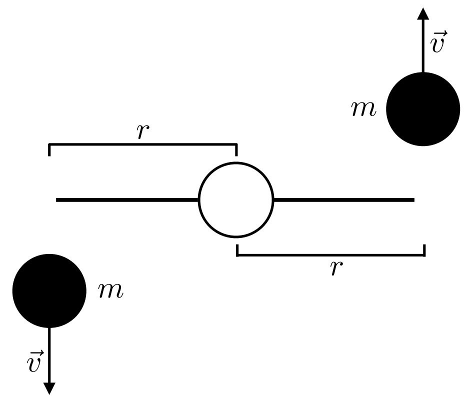
To analyze this problem, the fact that the weights fall due to gravity is completely a red herring and doesn't affect the result. So, let's instead imagine that we are just rotating in space, far from any stars. Before releasing the weights, I and the weights are rotating as



and our angular momentum is

$$L_{\text{tot}} = I_{\text{me}}\omega + 2mr^2\omega\,. \tag{9.23}$$

When I release the weights, they cease their circular motion and travel in a straight line tangent to the original circular motion



Note that the speed v is just $v = r\omega$, so that the angular momentum of the two masses with respect to my head (the axis of rotation) is

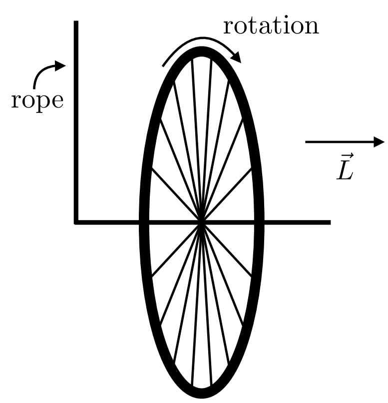
$$L_{\text{masses}} = 2|\vec{r} \times \vec{p}| = 2r(mv) = 2r(mr\omega) = 2mr^2\omega\,. \tag{9.24}$$

This is identical to before releasing the masses. Simply releasing the masses exerted no torque so necessarily angular momentum is conserved in this reaction. Therefore, if angular momentum is conserved and the angular momentum of the masses didn't change, then so too must my angular momentum remain the same.

Let's test this out! (See `https://youtu.be/g7QLKoMQGOM`)

9.2.2 Torques Applied Orthogonal to Angular Momentum

Okay, good enough, now let's analyze another rotating system. We're going to bring back our bicycle wheel and get it rotating. Once rotating, I'm going to only hold it by a rope attached to the end of one of the handles. A picture of this is



My question to you is: what happens? What does the wheel do once this system is released to the world? Possible answers include:

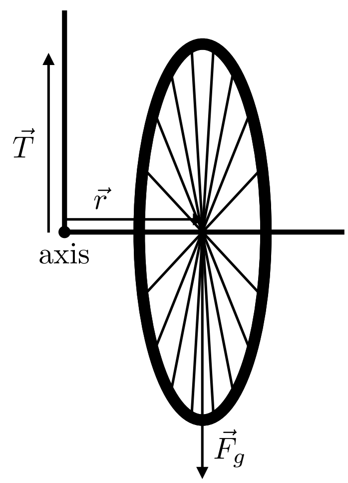
(a) Nothing, remains stationary

(b) Falls into a vertical position

(c) Revolves around rope

(d) Oscillates like a pendulum about end of rope

To answer this question, let's first consider what happens if the wheel were stationary and not rotating. In that case, we will analyze the torques of the system about an appropriate axis. Note that the end of the rope is stationary in this setup, so we will call the end of the rope the axis of rotation. Then, let's draw the forces on the wheel:



There is tension in the rope, and gravity acts at the center-of-mass of the wheel. The tension exerts no torque because it is applied at the axis of rotation. By contrast, gravity does exert a torque about the axis because it is displaced from the axis and an angle of 90° with respect to $\vec{r}$. By the right-hand rule, the direction of this torque is

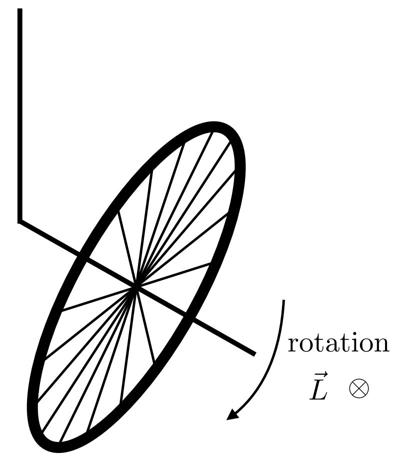
$$\vec{\tau}_g = \vec{r} \times \vec{F}_g \quad \otimes \,, \tag{9.25}$$

into the page, while the magnitude is

$$|\vec{\tau}_g| = rF_g = rmg \,, \tag{9.26}$$

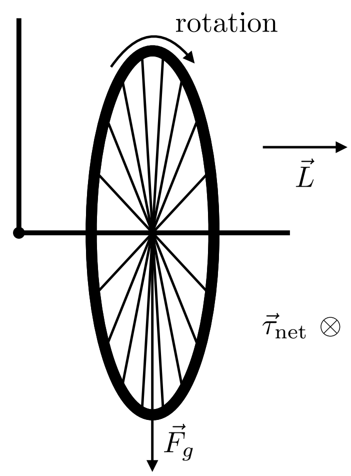
which is indeed non-zero.

Of course, we know what happens when we release the wheel in this case: it falls, swinging clockwise, and so has angular momentum into the page

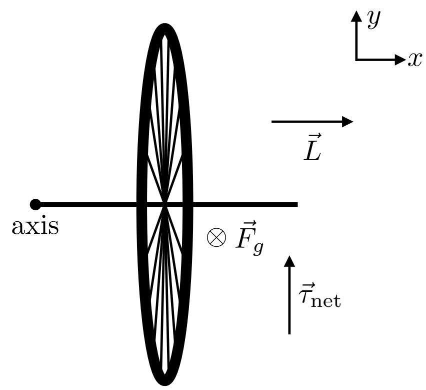


This is exactly as predicted by $\vec{\tau} = \frac{d\vec{L}}{dt}$.

Okay, that was easy. Now, let's get the wheel rotating and think about what happens. We have the same torque as in the non-rotating case



But now, we have a non-zero angular momentum, and that makes all the difference. Let's first consider what happens a very short time after we release the system to the wild. To do this, let's draw an overhead picture of the system which will clarify what's going on:



For argument's sake, let's call the initial angular momentum

$$\vec{L}_i = L\hat{\imath}\,, \tag{9.27}$$

and the direction of torque is then

$$\vec{\tau}_{\text{net}} = \tau\hat{\jmath}\,. \tag{9.28}$$

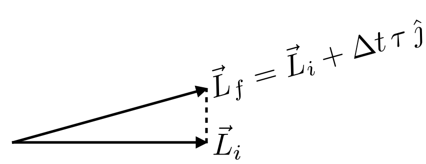
Then, by Newton's law for rotations, we have

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \Rightarrow \qquad \tau\hat{\jmath} \approx \frac{\vec{L}_f - \vec{L}_i}{\Delta t}\,, \tag{9.29}$$

where $\vec{L}_f$ is the angular momentum evaluated at a time Δt after I release the system. Solving for $\vec{L}_f$, we find that

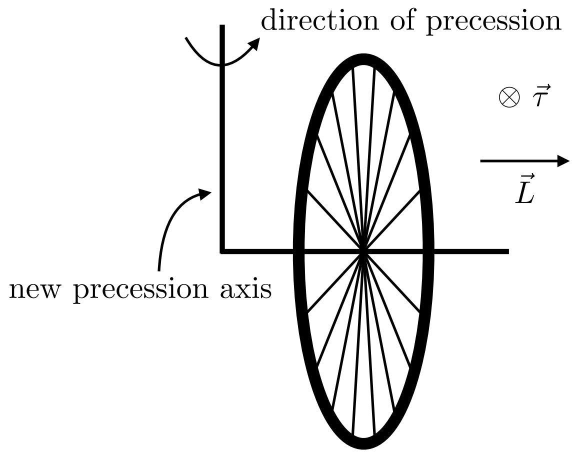
$$\vec{L}_f = L\hat{\imath} + \Delta t\,\tau\hat{\jmath}\,, \tag{9.30}$$

that is, the angular momentum picks up a y component. The picture of this is



So apparently the angular momentum just rotates about the rope! We can continue this to a later time by asking what happens to $\vec{L}_f$ because of the torque, etc. A rotating wheel does *not* fall: if it did, the direction of the change of angular momentum would be inconsistent with Newton's second law.

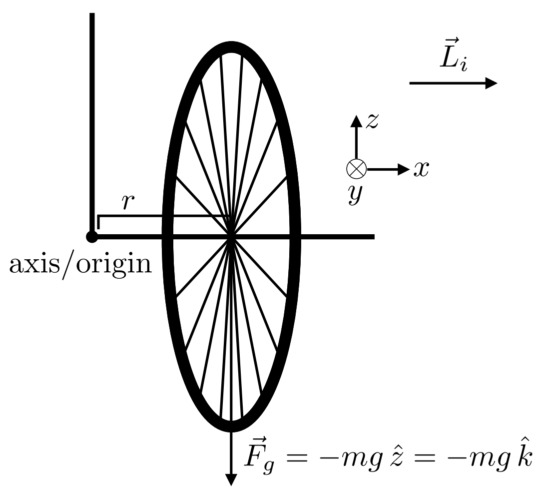
This extremely weird rotational phenomena is called **precession**. If a torque acts parallel (or anti-parallel) to angular momentum, then it acts to increase or decrease the angular velocity. However, if a torque acts perpendicular to the direction of angular momentum, as in the case at hand, it acts to precess, or rotate, the angular momentum's direction about the axis of rotation that is perpendicular to both the torque and angular momentum. In this case, we have



Let's see this in action! (See `https://youtu.be/cj_cC53IgXA`)

9.2.3 Precession as Circular Motion of Angular Momentum

Let's now study Newton's second law and see if we can massage it into an interesting form to solve for precessing angular momentum $\vec{L}$ valid at all times t. Let's start by putting down coordinates:



With this setup, note that the vector $\vec{r}$ from the origin/axis to the center of mass of the wheel is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}\,, \tag{9.31}$$

where r is the distance from the axis to the center of the wheel and θ is the angle the vector makes with respect to the x axis, in the xy plane. Then, the torque is

$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F}_g = -mg\,(r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}) \times \hat{k} \\ &= -rmg\,(\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath})\ .\end{aligned} \tag{9.32}$$

Note that the direction of angular momentum is radial, away from the origin, so we can express it as

$$\vec{L} = L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath}\,. \tag{9.33}$$

Because the torque acts perpendicular to angular momentum, the magnitude of angular momentum is fixed, just like the magnitude of linear momentum is fixed if a force acts perpendicular to momentum. Now, Newton's second law says that

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \Rightarrow \qquad (-rmg\sin\theta)\hat{\imath} + (rmg\cos\theta)\hat{\jmath} = L\frac{d\cos\theta}{dt}\,\hat{\imath} + L\frac{d\sin\theta}{dt}\,\hat{\jmath}\,. \tag{9.34}$$

Let's focus on the $\hat{\imath}$ components of this equation:

$$-rmg\sin\theta = L\frac{d\cos\theta}{dt}\ . \tag{9.35}$$

Using the chain rule, we have

$$L\frac{d\cos\theta}{dt} = L\frac{d\cos\theta}{d\theta}\frac{d\theta}{dt} = -L\omega\sin\theta = -rmg\sin\theta\,. \tag{9.36}$$

Canceling factors, we find that the angular velocity of precession is

$$\omega = \frac{rmg}{L}\,, \tag{9.37}$$

which is constant. We can do the same analysis for the $\hat{\jmath}$ component,

$$L\frac{d\sin\theta}{dt} = L\frac{d\sin\theta}{d\theta}\frac{d\theta}{dt} = L\omega\cos\theta = rmg\cos\theta\,, \tag{9.38}$$

or again, that

$$\omega = \frac{rmg}{L}\,. \tag{9.39}$$

Therefore, the angular momentum under this precession is

$$\vec{L} = L\cos\left(\frac{rmg}{L}t\right)\hat{\imath} + L\sin\left(\frac{rmg}{L}t\right)\hat{\jmath}\,. \tag{9.40}$$

Note that this sweeps out a circle over time, of radius L and at rate $\omega = \frac{rmg}{L}$. That is, precession is nothing more than uniform circular motion of angular momentum.

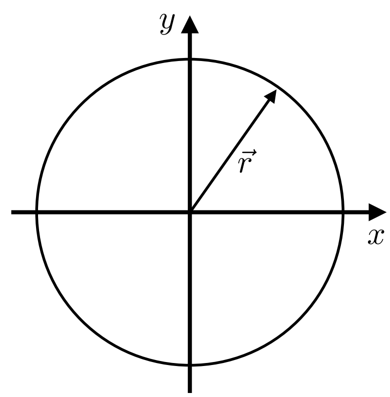
Chapter 10

Oscillations

In these final lectures, we are going to study the general phenomena of oscillations. We have encountered oscillations at several points throughout this course. First, when we introduced circular motion long ago, we had expressed the position vector $\vec{r}$ of an object undergoing circular motion as

$$\vec{r}(t) = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath}\,, \tag{10.1}$$

and this vector looks like



and it revolves counterclockwise at a rate of ω rad/s. The radius of the circle is r.

At the end of the previous chapter, we introduced precession, or the uniform circular motion of angular momentum due to a torque applied perpendicular to the direction of angular momentum. We had found that the resulting angular momentum vector $\vec{L}$ can be expressed as

$$\vec{L}(t) = L\cos\left(\frac{\tau}{L}t\right)\hat{\imath} + L\sin\left(\frac{\tau}{L}t\right)\hat{\jmath}\,, \tag{10.2}$$

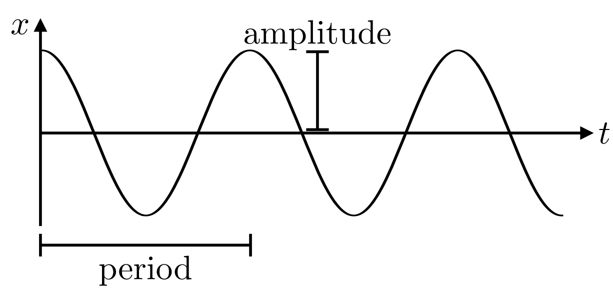
where L is the magnitude of the angular momentum (constant in time) and τ is the magnitude of the torque applied perpendicular to angular momentum. This precession or "uniform circular motion" of angular momentum is exactly analogous to the uniform circular motion described earlier. If a force is applied exclusively perpendicular to velocity/momentum, then no work is done on the object (no speed is changed) and only the direction of motion is affected. So, precession and uniform circular motion are very much so the same phenomena, just manifest in different systems.

10.1 Kinematics of Oscillations

In what sense, however, are either of these systems "oscillating," or swung back and forth (in Latin)? Instead of considering the circle swept out as a function of x and y position as time passes, let's just focus on one coordinate; say, the x-coordinate of uniform circular motion as a function of time. That is, let's just focus on the function

$$x(t) = r\cos\omega t\,. \tag{10.3}$$

What does this look like? Plotting it, we have

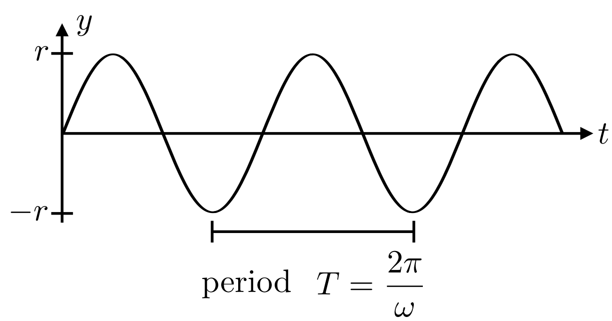


This position versus time graph illustrates clearly "oscillation" and the graph has a number of features that we give names to. First, the maximum distance of the oscillation away from the abscissa (horizontal axis) is called the **amplitude** of the oscillation. Because the maximum value of cosine is 1, the amplitude is simply the coefficient of $\cos\omega t$; in this case, the distance r. Additionally, this oscillation repeats its pattern over a given length of time. The minimal length of time over which it repeats is called the **period** of oscillation. Colloquially, we often call this oscillation a wave, but waves (properly) exhibit more phenomena than a generic oscillation. This oscillation that is controlled by a cosine (or sine) is referred to as a **sinusoidal oscillation**.

Okay, we've got our bearings. Now let's take a look at the y-component of uniform circular motion. We have the formula

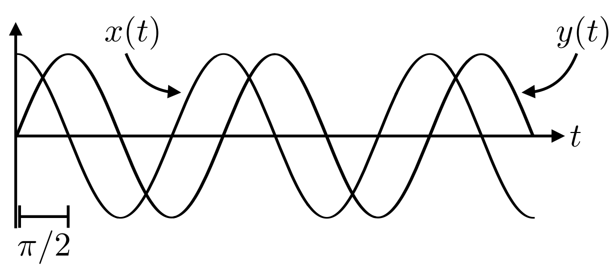
$$y(t) = r \sin \omega t \,, \tag{10.4}$$

and a plot of this as a function of time is



Note that this oscillation has the same period and amplitude as for the x coordinate. However, we say that $x(t)$ and $y(t)$ oscillations are 90° (or $\pi/2$ radians) out of phase because their values at $t = 0$ are different, by an angular factor of $\pi/2$.

To see this graphically, let's plot them on the same plot:



Over one period T, ωt increases by 2π

$$\omega(t + T) = \omega \left(t + \frac{2\pi}{\omega} \right) = \omega t + 2\pi \,, \tag{10.5}$$

and so we also say that a period corresponds to an angular displacement of 2π (like going all the way around a circle). Note that from $x(t)$ at $t = 0$, one has to travel one-quarter period in time to get to the same point of $y(t)$. One quarter period is $2\pi/4 = \pi/2$ radians, thus the $\pi/2$ out of phase.

Symbolically, let's manipulate $\sin \omega t$ into a form with $\cos \omega t$. To do this, note that for $y(t)$ to overlap $x(t)$, it needs to be moved left by a phase angle $\pi/2$. That is, at $t = 0$, it needs to have an argument that is smaller by $\pi/2$ that in $\sin \omega t$ form. This is just to say that

$$\sin \omega t = \cos\left(\omega t - \frac{\pi}{2}\right) . \tag{10.6}$$

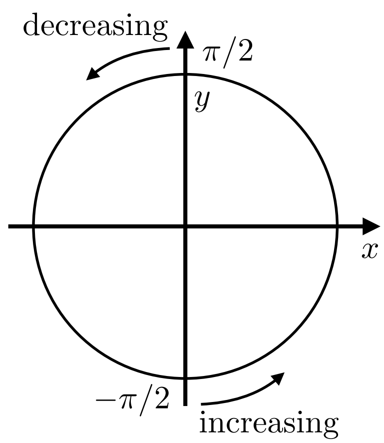
Let's see if this makes sense. First, at $t = 0$, we have

$$\sin 0 = \cos\left(0 - \frac{\pi}{2}\right) = 0 , \tag{10.7}$$

which is true. Now, regarding the "$-\pi/2$"; this is more subtle. In principle, we could have had the relationship

$$\sin \omega t \stackrel{?}{=} \cos\left(\omega t + \frac{\pi}{2}\right) , \tag{10.8}$$

and this would also have satisfied the requirement at $t = 0$. However, it doesn't work immediately after. Let's go back to our unit circle:



At $t = 0$, we have the two options: a phase of $\pi/2$ or $-\pi/2$, which I have illustrated. As time increases, we move around the unit circle in a counterclockwise manner. This means that, from phase point $\pi/2$, for example, the x-component (cosine) decreases, while from phase point $-\pi/2$ the x-component increases. Looking at the graph of $\sin \omega t$, as t increases, does $\sin \omega t$ increase or decrease? It increases, so therefore

$$\sin \omega t = \cos\left(\omega t - \frac{\pi}{2}\right) . \tag{10.9}$$

Another symbolic way to identify the same result is simply by taking the derivative and then setting $t = 0$. Note that

$$\frac{d}{dt}\sin\omega t\bigg|_{t=0} = \omega\cos\omega t|_{t=0} = \omega > 0\,, \tag{10.10}$$

and also that

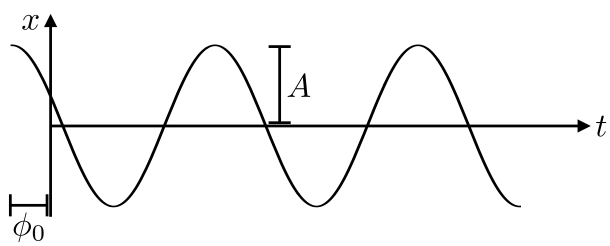
$$\frac{d}{dt}\cos\left(\omega t \pm \frac{\pi}{2}\right)\bigg|_{t=0} = -\omega\sin\left(\omega t \pm \frac{\pi}{2}\right)\bigg|_{t=0} = \mp\omega\,, \tag{10.11}$$

so for these to agree, we must take the "$-\pi/2$" phase factor.

Finally, I just want to note that a generic sinusoidal oscillation can be expressed as

$$x(t) = A\cos(\omega t + \phi_0)\,, \tag{10.12}$$

where A is the amplitude, ω is the angular frequency, and ϕ_0 is the phase:



Note that $+\phi_0$ means that the wave is shifted left. This general form of the oscillation follows from the angle addition formula

$$\cos(\omega t + \phi_0) = \cos\phi_0\,\cos\omega t - \sin\phi_0\,\sin\omega t\,. \tag{10.13}$$

We'll use this formalism to understand physical phenomena next section. To end this section, I want to ask a couple of questions.

Example

First, we had said that uniform circular motion is described by the vector

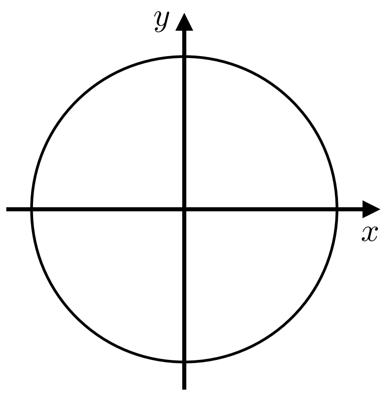
$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\sin\omega t\,\hat{\jmath}\,. \tag{10.14}$$

Note that the amplitudes of the two components are identical, r. What if they are different, with $r_1 > r_2$,

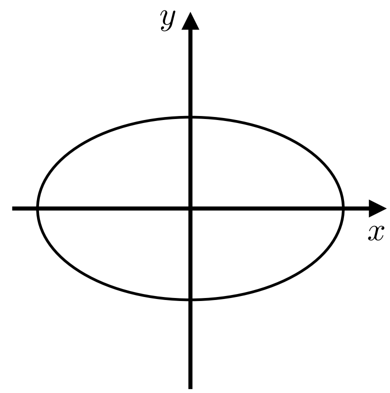
$$\vec{r}(t) = r_1 \cos \omega t \, \hat{\imath} + r_2 \sin \omega t \, \hat{\jmath} \, . \tag{10.15}$$

What does the trajectory of the object look like now?

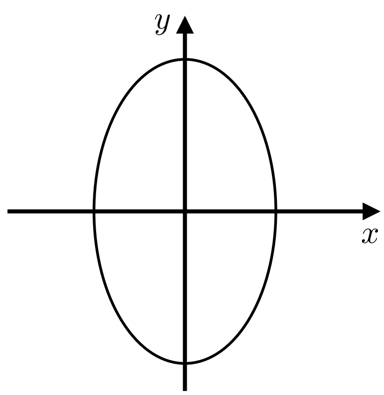
(a)



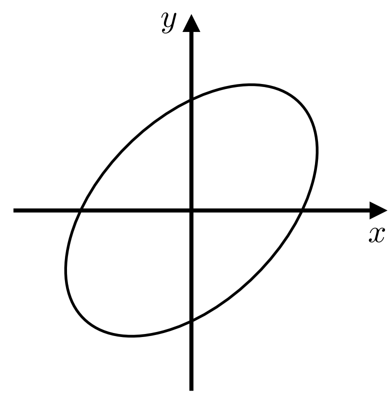
(c)



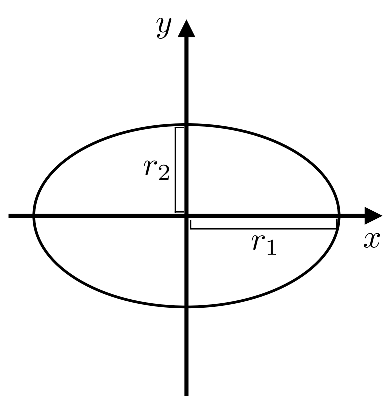
(b)



(d)



The answer is (c). Because $r_1 > r_2$, the trajectory goes farther from the origin in the x direction than in the y direction. Actually, this trajectory is nothing more than an ellipse with semimajor axis r_1 and semiminor axis r_2



Example

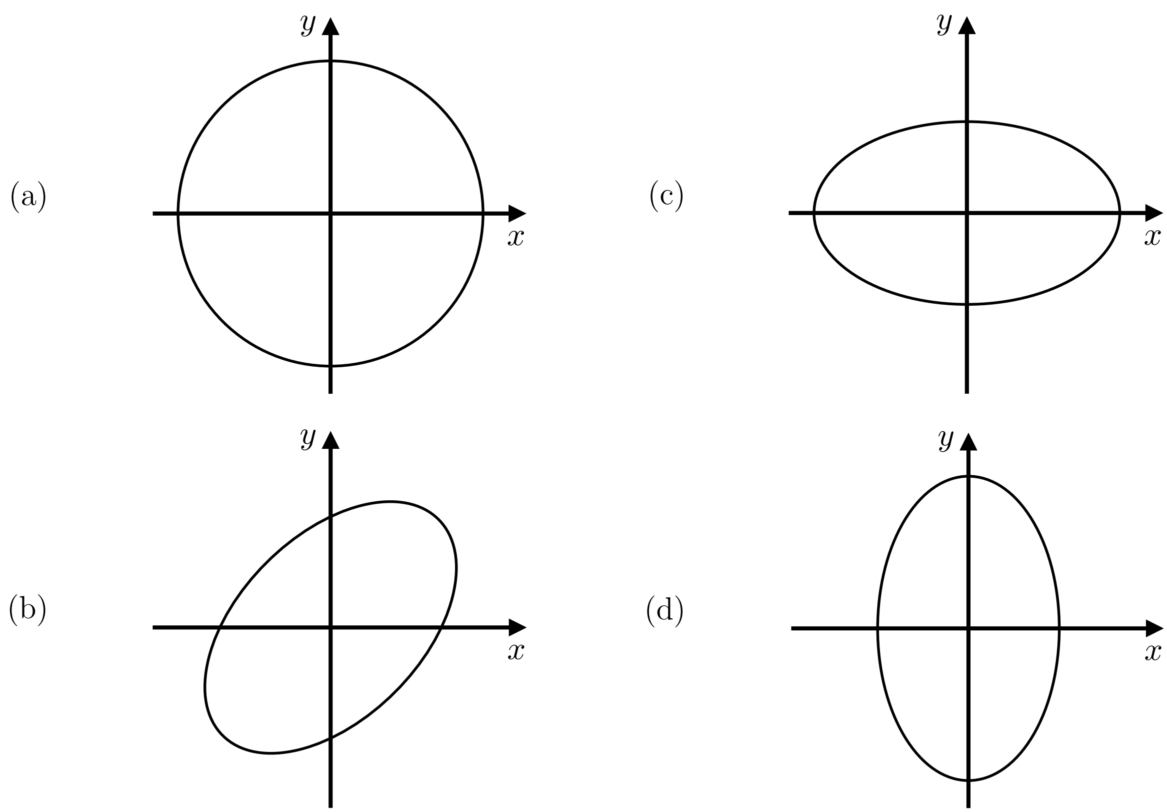
We had also said that $\sin\omega t$ is out of phase of $\cos\omega t$ so the vector for uniform circular motion is

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\cos\left(\omega t - \frac{\pi}{2}\right)\,\hat{\jmath}\,. \tag{10.16}$$

This out of phase–ness is vital to produce a circular trajectory. However, what if the phase difference were only $-\pi/4$, instead of $-\pi/2$? That is, what trajectory would the vector

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\cos\left(\omega t - \frac{\pi}{4}\right)\,\hat{\jmath}\,, \tag{10.17}$$

sweep out?



The answer is (b). This is very tricky, but one way to see it is to evaluate $\vec{r}(t)$ at $\omega t = 0, \pi/2, \pi, 3\pi/2$ and connect the dots. Decreasing the phase difference between the x- and y-components rotates and smushes the trajectory. Consider what happens if that $-\pi/4$ is turned into 0. What is the trajectory now?

10.2 Simple Harmonic Oscillators

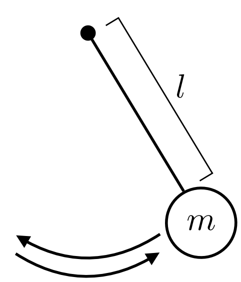
Previously, we introduced the language of oscillations, specifically sinusoidal oscillations. We had said that an oscillating system can, in general, be expressed as

$$x(t) = A\cos(\omega t + \phi_0)\,, \tag{10.18}$$

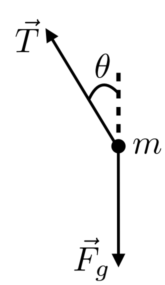
where A is the amplitude of oscillation, ω is the angular frequency of oscillation, and ϕ_0 is called the phase (factor) of the oscillation, which just quantifies where the object starts at $t = 0$. Here, $x(t)$ denotes anything that might oscillate: position, angular momentum, angle, etc., and a different physical system will have a different quantity that oscillates. In this lecture, we will use this general formula to analyze some familiar physical systems.

10.2.1 The Pendulum

We'll start with the pendulum, which as I have illustrated before, is nicely exhibited with a bowling ball attached to the end of a hanging rope. Let's remind ourselves of what this pendulum is doing. I release the pendulum from some initial position and then it swings back and forth, that is, oscillates, like so:



On this figure, I have denoted the mass of the bowling ball as m and the length of the rope as l. To analyze the oscillation of the pendulum, let's draw the free-body diagram of the mass



$\vec{T}$ is the tension in the rope and $\vec{F}_g = mg\hat{\jmath}$ is the gravitational force.

As the mass oscillates, it remains a fixed distance l from the axis of oscillation. That is, the mass sweeps out the arc of a circle as it swings back and forth. As such, there is no motion perpendicular to this circle, so we know that forces that act in the direction of the axis of oscillation, i.e., centripetal forces, act only to change the direction of velocity of the mass.

So, the relevant force for oscillation is the force tangent to the circular arc, that is responsible for the ball speeding up and slowing down while it oscillates. What is this force? Well, the tension is exclusively centripetal, so not that. The only relevant force then is a component of gravity that depends on the angle of the rope with respect to vertical:

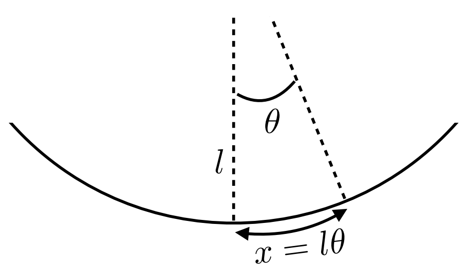
$$F_{\mathrm{rel}} = -mg\sin\theta\,. \tag{10.19}$$

Note the lack of a vector: we can consider the oscillations as one-dimensional. They are just fixed to the arc of a circle. Also notice the "$-$" sign: this is a restoring force acting to always pull the ball toward $\theta = 0$ where the rope is vertical.

Now that we have this force, we want to use it to determine Newton's second law. Newton's second law is of course

$$F = ma = m\,\frac{d^2x}{dt^2}\,, \tag{10.20}$$

but we need to figure out what the position x is. Recall that the motion of the pendulum is along the arc of a circle, so the distance x is just some distance measured along the arc of this circle:



Now, let's take derivatives of $x = l\theta$:

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(l\theta) = l\frac{d^2\theta}{dt^2} = l\ddot{\theta}\,. \tag{10.21}$$

The length of the rope l is constant, so it just pulls out of the derivative. Then, our Newton's second law is

$$-mg\sin\theta = ml\ddot{\theta}\,. \tag{10.22}$$

This is a differential equation for the angle θ, the angular dependence from vertical as a function of time. It is called a second-order differential equation because there are at most two derivatives of θ present in the equation. Further, it is a non-linear differential equation because a non-linear function of θ, namely $\sin\theta$, is present in the equation. As a non-linear second-order differential equation it is extremely challenging to solve (see the Wikipedia page on "Pendulum (Mathematics)" for details).

So it might seem like we are up a creek without a paddle as they say, and can't go further. If we only consider small angular displacements from vertical, that is, $\theta \ll 1$ in radians, then the $\sin\theta$ non-linear function can be simplified via Taylor expansion. For $\theta \ll 1$, $\sin\theta$ is approximately

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots \approx \theta\,, \tag{10.23}$$

which is a linear function of θ. With this assumption, our Newton's second law becomes

$$-mg\theta = ml\ddot{\theta}\,, \tag{10.24}$$

which is simple and can easily be solved. As we are studying oscillations, well, simple harmonic oscillators, we will just make the ansatz that the solution to this differential equation can be written as

$$\theta = A\cos(\omega t + \phi_0)\,, \tag{10.25}$$

for some amplitude A, angular frequency ω, and phase ϕ_0. Let's just plug this into the differential equation and see what we find.

The second derivative of θ is:

$$\frac{d^2}{dt^2}A\cos(\omega t + \phi_0) = A\frac{d}{dt}\left(-\omega\sin(\omega t + \phi_0)\right) = -A\omega^2\cos(\omega t + \phi_0)\,, \tag{10.26}$$

so Newton's second law is

$$-mg\left(A\cos(\omega t + \phi_0)\right) = -mAl\omega^2\cos(\omega t + \phi_0)\,, \tag{10.27}$$

or, canceling factors,

$$g = l\omega^2 . \tag{10.28}$$

Note the miracle that happened here. We initially had a second-order differential equation we had to solve. With our sinusoidal ansatz, we transmogrified the differential equation into a second-order algebraic equation that is trivial to solve:

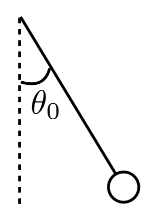
$$\omega = \sqrt{\frac{g}{l}} . \tag{10.29}$$

This general procedure of transforming a differential equation into an algebraic equation with a sinusoidal ansatz is called a **Fourier transform**, after Joseph Fourier, a 19th century French mathematician. Fourier, among other things, first described the greenhouse effect.

So this Fourier transform immediately gives us the angular frequency, so our solution for the angle as a function of time is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{l}}t + \phi_0\right) . \tag{10.30}$$

What about A and ϕ_0? The differential equation can't help us there, but the initial conditions of the pendulum can. Initially, if we hold the pendulum an angle θ_0 from vertical:



by energy conservation, we know that at any later time, the angle of the pendulum can never be larger than θ_0. That is, θ_0 is the amplitude of oscillation, and this amplitude is the angular displacement at $t = 0$. Thus, $\phi_0 = 0$ and

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{l}}t\right) . \tag{10.31}$$

Whew!

There are a few interesting things to note about this result:

- The velocity of the pendulum is

$$v(t) = l\frac{d\theta}{dt} = -\theta_0\sqrt{gl}\sin\left(\sqrt{\frac{g}{l}}t\right) . \tag{10.32}$$

Note that the velocity is 90° ($= \pi/2$) out of phase with the position of the pendulum.

- We could have guessed $\omega \propto \sqrt{g/l}$ by dimensional analysis. In the statement of the problem, the only relevant dimensionful quantities were acceleration g, rope length l, and mass m. The only way that these can combine into units of frequency (time^{-1}) is $\sqrt{g/l}$.

- There is a bit of numerology regarding the history of the pendulum and SI units. Note that the frequency f is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} , \tag{10.33}$$

and so the period of the pendulum is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{l}{g}} . \tag{10.34}$$

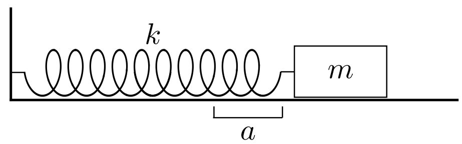
Why $g = 9.8$ m/s^2? Well, note that the period of the pendulum is independent of amplitude θ_0, so a pendulum of length l is a very good, regular keeper of time. This has been known for at least 400 years. The length l is very easy to measure, but g is more subtle. Wouldn't it be very convenient if the hard thing to measure, g, canceled with the hard number to express, π? Indeed, note that

$$\pi^2 = 9.8696\cdots , \tag{10.35}$$

eerily close to the SI value of g!

10.2.2 The Spring

To end this lecture, I want to briefly introduce the spring simple harmonic oscillator



We have a mass m connected to a spring with constant k, initially displaced from equilibrium by a distance a. My first question to you is: using dimensional analysis, what is the angular frequency ω of the mass as it oscillates?

(a) $\omega = \sqrt{ak}$ (b) $\omega = \sqrt{\frac{k}{m}}$ (c) $\omega = \sqrt{\frac{m}{k}}$ (d) $\omega = a\sqrt{km}$

Let's solve Newton's second law and see what we find! The spring force, Hooke's law, is

$$F = -kx\,, \tag{10.36}$$

and so Newton's second law is

$$-kx = m\ddot{x}\,. \tag{10.37}$$

This is almost exactly similar to Newton's second law for the pendulum! So, let's use our ansatz for simple harmonic oscillation to solve this:

$$x(t) = A\cos\left(\omega t + \phi_0\right)\,, \qquad \ddot{x}(t) = -\omega^2 A\cos\left(\omega t + \phi_0\right)\,. \tag{10.38}$$

Plugging this into Newton's second law, we have

$$-kA\cos\left(\omega t + \phi_0\right) = -m\omega^2 A\cos\left(\omega t + \phi_0\right)\,, \tag{10.39}$$

or that

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{10.40}$$

Further, if the spring is initially stretched by length a, conservation of energy states that its displacement from equilibrium can never be larger than a. Therefore, the amplitude is a and $\phi_0 = 0$ and so

$$x(t) = a\cos\left(\sqrt{\frac{k}{m}}t\right)\,. \tag{10.41}$$

10.3 Two Other Topics on Oscillations

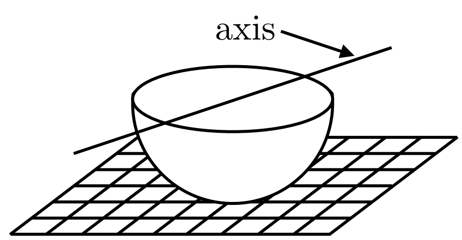
In this final lecture, we are going to wrap up a couple of loose ends, extending and connecting some earlier lectures and just introducing **traveling waves**. As we have for the previous several lectures, we continue studying oscillations, as modeled by the sinusoidal function

$$x(t) = A\cos(\omega t + \phi_0)\,, \tag{10.42}$$

where A is the amplitude, ω is the angular frequency, and ϕ_0 is the phase of oscillation. We've seen this formula applied in numerous cases already, and here we will end with two more.

10.3.1 Physical Pendula

The first system we will study is that of a physical pendulum, a realistic object that oscillates about an axis. The object we will consider here is a solid hemisphere, and we are asked to identify the period of oscillation when set upon its rounded end and slightly perturbed:



To be able to solve this problem, we will need to know both the center-of-mass of the hemisphere, as well as its moment of inertia about the axis that sits on the flat part (as illustrated).

Let's first find the center-of-mass. Let's call the total mass of the hemisphere M and note that because it is, um, half of a sphere, its volume is

$$\text{Vol} = \frac{1}{2}\text{Vol}_{\text{sphere}} = \frac{2}{3}\pi R^3\,, \tag{10.43}$$

where R is its radius. Then, the density of the hemisphere is

$$\rho = \frac{M}{\text{Vol}} = \frac{3}{2}\frac{M}{\pi R^3}\,. \tag{10.44}$$

Recall that the position vector of the center-of-mass of an object is

$$\vec{r}_{\rm cm} = \frac{1}{M} \int \vec{r}\, dm\,, \tag{10.45}$$

where dm is a small mass and $\vec{r}$ is the position vector of that small mass. Using the fact that we can write the small mass as

$$dm = \rho\, dV\,, \tag{10.46}$$

where dV is a small volume, the center-of-mass is

$$\vec{r}_{\rm cm} = \frac{\rho}{M} \int \vec{r}\, dV = \frac{1}{\rm Vol} \int \vec{r}\, dV\,. \tag{10.47}$$

Now, a while ago we had identified the best or easiest way to analyze a sphere; that is, by expression the small volume in spherical coordinates. Then, we had found that

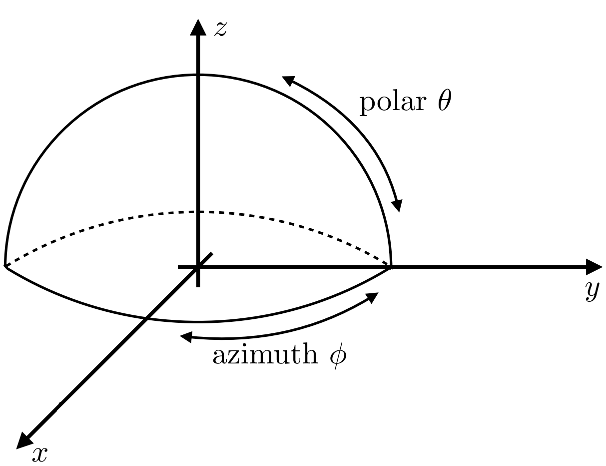
$$dV = r^2\, \sin\theta\, dr\, d\theta\, d\phi\,, \tag{10.48}$$

where r is the radial coordinate, θ is the polar angle, and ϕ is the azimuthal angle. So, for our hemisphere, we just need to calculate

$$\vec{r}_{\rm cm} = \frac{1}{\rm Vol} \iiint \vec{r}\, r^2\, \sin\theta\, dr\, d\theta\, d\phi\,. \tag{10.49}$$

This requires two things: figuring out what $\vec{r}$ is, and what the r, θ, and ϕ bounds of integration are.

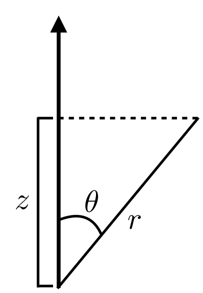
Let's draw the hemisphere more suggestively as



Recall that the polar angle for a sphere (lines of latitude) range over $\theta \in [0, \pi]$, but for a hemisphere, we only get from the North Pole to the equator. So, for a hemisphere, $\theta \in [0, \pi/2]$. The azimuthal angle still ranges over $\phi \in [0, 2\pi)$ because for every θ, we still can go all the way around a circle. Similarly, $r \in [0, R]$, the overall radius of the sphere. So, the center-of-mass calculation becomes

$$\vec{r}_{\rm cm} = \frac{1}{\rm Vol} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \vec{r}\, r^2 \, \sin\theta \, dr \, d\theta \, d\phi \,. \tag{10.50}$$

What is the vector $\vec{r}$? Well, by the illustration earlier, we immediately see that the x- and y- components of the center-of-mass vector $\vec{r}_{\rm cm}$ are 0. The only challenging component to evaluate is the z-component. Let's consider the illustration:



So, we see that $z = r\cos\theta$. Plugging this into the expression for the center-of-mass, we see that

$$\begin{aligned} z_{\rm cm} &= \frac{1}{\rm Vol} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R (r\cos\theta)\, r^2 \, \sin\theta \, dr \, d\theta \, d\phi \\ &= \frac{1}{\rm Vol} \left(\int_0^R r^3 \, dr \right) \left(\int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) . \end{aligned} \tag{10.51}$$

We can then do these integrals one at a time. The r and ϕ integrals are easy:

$$\left(\int_0^R r^3 \, dr \right) \left(\int_0^{2\pi} d\phi \right) = \frac{R^4}{4} \cdot 2\pi = \pi \, \frac{R^4}{2} \,. \tag{10.52}$$

We can make a u-substitution for the θ integral. If we set

$$u = \cos\theta \,, \tag{10.53}$$

then the θ integral becomes

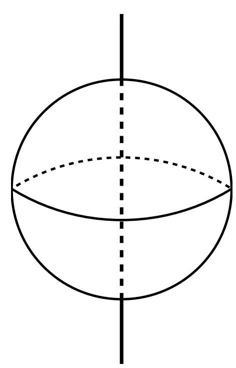
$$\int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta = \int_0^1 u \, du = \frac{1}{2}\,, \tag{10.54}$$

very simple! Putting it all together, we have

$$z_{\rm cm} = \frac{3}{2}\frac{1}{\pi R^3}\,\pi\frac{R^4}{2}\,\frac{1}{2} = \frac{3}{8}R\,. \tag{10.55}$$

That is, the z-coordinate of the center-of-mass is a distance of $\frac{3}{8}R$ from the flat side of the hemisphere. Does that make sense?

Now, on to calculating the moment of inertia. This is actually very simple with an observation. Let's draw a sphere with an axis of rotation that passes through its center:



Let's say that this sphere has radius R and mass $2M$. In this case, we know the moment of inertia is

$$I_{\rm sphere} = \frac{2}{5}(2M)R^2\,. \tag{10.56}$$

Now for the tricky bit. Let's imagine that this sphere is composed of two hemispheres, each of mass M, such that the axis of rotation lies on their flat faces. Then, it is clear that

$$I_{\rm sphere} = 2I_{\rm hemisphere} = 2\cdot\frac{2}{5}MR^2\,, \tag{10.57}$$

or that the hemisphere moment of inertia is simply

$$I_{\rm hemisphere} = \frac{2}{5}MR^2\,. \tag{10.58}$$

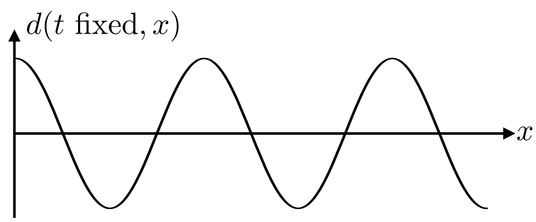
10.3.2 Traveling Waves

As a final topic in this course, we will but briefly introduce traveling waves. We'll start this by consideration of waves on a string. First and foremost, there is no net motion of the string, even though it appears that the string, or the wave on it is moving. What is happening is that each part of the string is oscillating up and down, but in a way that makes motion to the right appear, but actually doesn't happen.

So, if this string-wave is just oscillation of some flavor, we must be able to express the vertical displacement of any part of the string as

$$d(t) = A\cos(\omega t + \phi_0)\,, \tag{10.59}$$

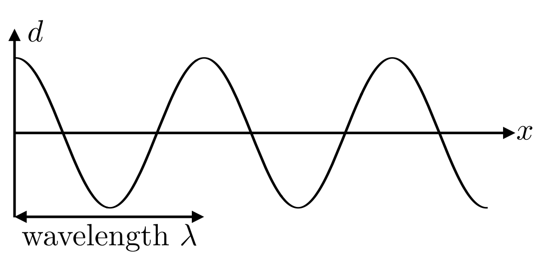
where A is the amplitude of displacement and ω is the angular frequency. This is simply related to how fast I move my hand up and down to create the wave. What is the phase ϕ_0? To answer this, let's consider a snapshot of the wave at a fixed time t. Then, the displacement as a function along the rope is



Because we fix time to, say, $t = t_0$, ωt is constant. Therefore, the only way that the displacement can vary as a function of position is if the phase ϕ_0 depends on position

$$\phi_0 \equiv \phi_0(x)\,. \tag{10.60}$$

What are the properties of this phase and its dependence on x? Note that the wave repeats itself after a minimal distance. We call the minimal distance over which the wave repeats itself the **wavelength** λ



Compare this to the definition of the period, T. Therefore, if the functional form of the displacement repeats every wavelength, the position-dependent phase must satisfy

$$\phi_0(x+\lambda) = \phi_0(x) + 2\pi\,. \tag{10.61}$$

The simplest way to do this is if

$$\phi_0(x) = \frac{2\pi}{\lambda}\,x \equiv kx\,. \tag{10.62}$$

Here, we call k the **wave number**, and it has a form similar to that of angular velocity,

$$\omega = \frac{2\pi}{T}\,. \tag{10.63}$$

Then, we have that the displacement of the rope as a function of time and position is

$$d(t,x) = A\cos\left(\omega t - kx\right)\,. \tag{10.64}$$

The relative "$-$" sign is because the wave travels to the right. Compare this with our identification of ϕ_0 for simple harmonic oscillation.

So, where does this perceived motion of the wave come from? Well, over time T, the rope's displacement repeats itself and in that time, the wave repeated itself over a distance λ. Then, the speed of the wave, that is, how fast the bumps appear to move right is

$$v_{\text{wave}} = \frac{\lambda}{T} = \frac{\omega}{k} = v_{\text{phase}}\,. \tag{10.65}$$

This particular wave speed is called the **phase velocity**. If we are able to express the angular frequency ω as a function of the wave number k, then we can define another speed called the **group velocity**,

$$v_{\text{group}} = \frac{d\omega}{dk}\,. \tag{10.66}$$

As a final mystery, recall that the kinetic energy of an object is

$$E = \frac{1}{2}mv^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m}\,, \tag{10.67}$$

related to momentum p. The speed of the object is

$$v = \frac{d}{dp}\frac{p^2}{2m} = \frac{2p}{2m} = \frac{p}{m} = v\,. \tag{10.68}$$

Fascinating! Is energy related to angular velocity ω and momentum to wave number k?

Classical Mechanics Glossary

acceleration: the rate of change of velocity with respect to time; the rate of change of the rate of change of position

amplitude: the maximum displacement of a wave or oscillation from equilibrium

angular momentum: a quantity of a massive object associated with its rotation about a fixed axis; if there are no torques on the object, angular momentum is conserved

angular velocity: the rate of change of an object's angle about a fixed axis with respect to time

azimuthal angle: the angle about a fixed axis ranging from 0 to 2π

center-of-mass: the position of a collection of masses corresponding to the mean location of the total mass; the point at which the net gravitational force acts for an extended object

centripetal acceleration: acceleration due to the change in an object's direction of velocity

chain rule: the rule for differentiating a composition of functions, where the result is the product of derivatives of each function in the composition

circular motion: motion of an object that travels in a circle

closed system: a system that is completely and perfectly isolated from any external environment

coefficient of static friction: the effective relative strength of the force of static friction as a fraction of the normal force exerted by a surface

conservation: the property enjoyed by a quantity that remains unchanged through time

conservative force: a force whose action is a direct consequence of a change in potential energy

cross-sectional area: the intersectional area of a plane that slices through a three-dimensional object; the area of an object's shadow

density: mass per unit volume

dimensional analysis: a technique for estimation from construction of a result from relevant quantities requiring that the units are correct

displacement: a vector of the change of position of an object with respect to a given origin

dot product: the scalar product of two vectors that is proportional to the cosine of the angle between them

elastic collision: a collision of two objects in which their net kinetic energy is conserved; a collision in which no non-conservative forces are present

electron-Volt: a unit of energy; the energy gained by an electron that travels through an electric potential of one Volt, approximately equal to 1.6×10^{-19} Joules

energy: the conserved quantity that measures a system's ability to perform a task

equivalence principle: the fundamental assumption of Newton's theory of universal gravitation that gravitational mass is equivalent to inertial mass

escape velocity: the minimal velocity that an object needs to completely leave the gravitational force of another object

force: an action that induces a change of velocity or momentum of an object

Fourier transform: the decomposition of a response into a linear combination of waves with different frequencies

frame of reference: the natural coordinate system for an object at rest

free-body diagram: a representation of the forces on an object that act directly on its center-of-mass

free fall: motion exclusively under the influence of gravity

friction: the force induced by two surfaces rubbing against one another

gedankenexperiment: German for "thought experiment", a technique for establishing physical consequences from experience and intuition alone

gravitational mass: the property of an object that is proportional to the gravitational force exerted on that object

group velocity: the velocity of a "group" of objects; the derivative of the angular frequency with respect to the wave number

Hooke's law: the restoring force law that governs a spring; a force that is proportional to opposite of the displacement from equilibrium

impact parameter: the distance of closest approach of a particle to its axis of rotation

impulse: force accumulated over time; the change in momentum

inelastic collision: a collision of two objects in which kinetic energy is not conserved; a collision in which non-conservative forces are important

inertial mass: the property of an object to oppose changes in motion; the quantity that appears in Newton's second law

inverse square law: a force that decreases inversely proportional to the square of the distance between objects

Joule: the SI unit of energy; kilogram–squared meters–per squared seconds

kinetic energy: the energy of motion of an object

lever arm: the perpendicular distance between the point of application of a force and the position with respect to an identified axis of rotation

moment of inertia: the property of an object to oppose changes in rotational motion of an object about an axis

momentum: a measure of the linear motion of an object; momentum is conserved if there are no forces acting on the object

Newton's constant: the constant of proportionality between gravitational mass and gravitational force

Newton's second law: the net force exerted on an object equals the object's mass times its acceleration, or its change in momentum with respect to time

Noether's theorem: the statement that conserved quantities have corresponding symmetries of the laws of physics, and vice-versa

non-conservative force: a force that cannot be expressed as the change of an appropriate potential energy

normal force: a force exerted normal or perpendicular to a surface

open system: a system that interacts with its surrounding environment

order-of-magnitude estimation: an estimation technique that employs a product of educated guesses for the numerical size of relevant quantities

pendulum: a weight hung from a rope or rod that swings under the influence of gravity

period: the minimal time over which a wave repeats itself

phase velocity: the velocity of an individual wave; the ratio of the angular velocity to the wave number

polar angle: the angle on a sphere measured with respect to one of the poles that ranges from 0 to π

potential energy: the energy of an object that is stored for future use

power: the rate of expending energy per time

precession: the phenomena of rotation of angular momentum about an axis due to a torque applied perpendicular to the initial angular momentum

projectile motion: motion of an object with a given initial velocity exclusively under the influence of gravity

range formula: the expression for calculation of the horizontal distance of projectile motion

reduced mass: the effective mass that orbits the center-of-mass in a gravitationally-bound system

restoring force: a force that is proportional to the opposite of displacement from equilibrium

right-hand rule: the rule or prescription for defining angular velocity, angular momentum, or torque from the direction of rotation or applied force with respect to an axis

rotational invariance: the property of a system that remains unchanged if it is rotated by any angle

scalar: a quantity that remains unchanged under rotation

simple harmonic oscillator: a system described by Hooke's law; a model of a mass attached to a perfect spring

sinusoidal oscillation: an oscillation described by sinusoidal motion about equilibrium

spatial translation: movement through space

spring constant: the constant of proportionality between displacement from equilibrium and force

static equilibrium: a state that exhibits no motion throughout time; a system on which no net forces nor torques are exerted

tangent: a line that intersects a curve at a single point that has the same slope as that point on the curve

Taylor series: the expansion of a function as represented by a polynomial of arbitrary order

theory of general relativity: the theory of gravitation introduced by Einstein that subsumes Newton's universal gravitation that describes gravity as the curvature of space and time

time-translation symmetry: the property of a system if it remains unchanged by moving through time

torque: an action that changes the angular momentum of an object

traveling waves: a wave that physically moves from an initiating point to a displaced location

unit vector: a vector of length 1 (unity) that exclusively encodes direction

universal law of gravitation: the theory due to Newton that describes the force of gravitation between two massive objects as linear in their masses and inversely proportional to the square of the distance between the objects

vector: a quantity that encodes magnitude and direction in multiple dimensions

vector addition: the method for adding together two vectors in which each component of the vectors are summed

vector cross product: the product of two vectors that returns another vector; the resulting direction from the vector cross product is determined by the right hand rule

velocity: the rate of change of displacement per time

wavelength: the minimal distance over which a wave repeats itself

wave number: 2π divided by the wavelength

weight: the force of gravity near the surface of Earth that represents the minimal force that one must exert to lift the object

work: force accumulated over distance; the change in kinetic energy

Work-Energy Theorem: the equality of work to change in mechanical energy