Lecture 14 Physics 101

Welcome to Friday! Please turn in Homework, and remember that there is an exam (closed book, closed notes) one week from today.

Last lecture we had introduced energy, its manifestation as moving or <u>kinetic</u> <u>energy</u>, and the work-energy theorem. The ~~the~~ work-energy theorem is the Statement that the change in kinetic energy of an object is the amount of work done on that object:

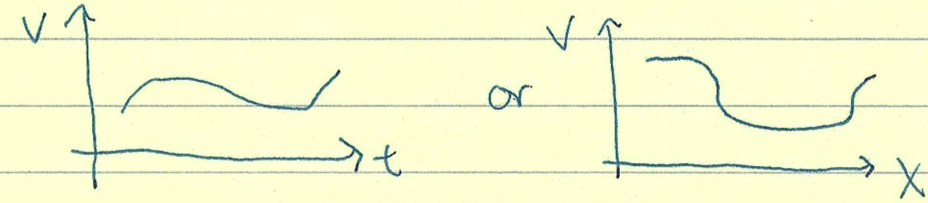
$$\Delta K = W.$$

We had introduced kinetic energy as

$$K = \frac{1}{2}mv^2 \text{, and so } \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2,$$

for initial and final velocities, v_i and v_f, respectively. We had briefly introduced work, but we will make it more precise in this lecture.

Let's first consider motion in one-dimension. Let's say that we have an object of mass m which is being acted on by a force F. This mass therefore has a velocity that varies as a function of time or position:



like so. We would like to derive a relationship of the change in kinetic energy ΔK of the object as it travels to the force acting on it.

The difference in kinetic energy at times t and $t+\Delta t$ is:

$$\Delta K = \frac{1}{2} m v(t+\Delta t)^2 - \frac{1}{2} m v(t)^2$$

As $\Delta t \to 0$, this can be related to the time derivative of kinetic energy:

$$\frac{dK}{dt} = \lim_{\Delta t \to 0} \frac{\frac{1}{2} m v(t+\Delta t)^2 - \frac{1}{2} m v(t)^2}{\Delta t}$$

$$= m v a,$$

where $a = \frac{dv}{dt}$, and we used the chain rule. Note that $v(t)$ is velocity at time t, not velocity multiplied by t. Now, using Newton's second law, note that

$$m v a = v F = F \frac{dx}{dt},$$ where on the right, we just note that velocity is the time derivative of position x. Thus, simply differentiating kinetic energy with time, we have found:

$$\frac{dK}{dt} = F \frac{dx}{dt}.$$

Now for some trickery... In this Leibniz notation, derivatives are, really I swear, ratios, so we can cross cancel and divide. So, we "cancel" the dt factors, and find

$dK = F\,dx$, or dividing by dx on both sides

$\frac{dK}{dx} = F$. That is, the derivative of kinetic energy with respect to position x is Force!

Almost there: let's integrate both sides of this expression over position x from $x=a$ to $x=b$. The Fundamental Theorem of Calculus states that:

$$\int_a^b \frac{dK}{dx}\,dx = K(x=b) - K(x=a) \equiv \Delta K$$

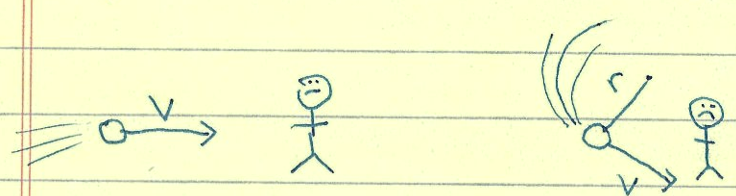
while integrating over the force, we find

$$\Delta K = \int_a^b F\,dx.$$

This is the work-energy theorem: the change in kinetic energy of an object is equal to the integral of force over the trajectory of the object. This implies Newton's second law and vice-versa.

That's the story in one-dimension; how do we generalize the work-energy theorem to multiple dimension motion? Let's think again about what kinetic energy is and how forces can change it.

Now, I want to introduce a definition of kinetic energy that will serve our purposes for considering the generalization of work-energy theorem. I will define kinetic energy of a ball, for example, as measured by how much it hurts when it hits you. Let's consider two set-ups: one where the ball is thrown straight at you with speed v and the other where the ball is attached to a string and rotated such that the ball has tangential speed v. That is, we have the setups:



Which ball will hurt more when it hits you?

a) linear ball b) circular ball c) same hurt

I'll give you a couple minutes! Okay, now can I get a couple of volunteers? Just kidding :)

Immediatly before the balls hit you they were both traveling with speed v. Who cares that one ball was traveling in a line and the other in a circle: all of that velocity, for both balls, has to stop when they hit you. So they will both hurt the same!

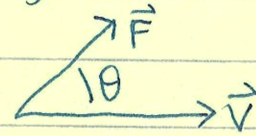
Why is this relevant? Well, are the accelerations of the ball's different? What would that suggest about the forces acting on them? Correspondingly, what would that naiively suggest for the work-energy theorem in multiple dimensions?

For the ball moving in a line, it's kinetic energy is constant, $\frac{1}{2}mv^2$, until it hits you. There are no forces on the ball: its velocity vector $\vec{v}$ is constant. For the ball moving in the circle, its velocity vector $\vec{v}$ is continuously changing direction, but keeps its speed constant. Apparently simply changing direction does not change kinetic energy. That is, accelerations and therefore forces that exclusively change the direction of velocity do no work.

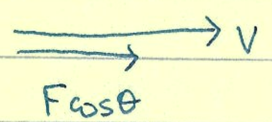
What's special about the acceleration that keeps the ball moving in a circle? It is centripetal acceleration, and as we discussed earlier is perpendicular to tangential velocity. That is, the force responsible for keeping the ball in a circle is perpendicular to the motion of the ball. This identification suggests that forces exerted perpendicular to motion only change direction, and do no work.

Therefore, to determine the work done on an object, we only care about those forces with components in the direction of motion. Only they can do non-zero work.

So let's analyze this for a particular velocity vector and force. Let's say we exert a force $\vec{F}$ on an object of mass m with velocity $\vec{v}$ as:



Let's call the angle between the force and velocity θ. As argued earlier, the component of $\vec{F}$ perpendicular to $\vec{v}$ does no work, so just for studying energy, this is equivalent to:



or that $\frac{dK}{dx} = F\cos\theta$.

If we turn infinitesimals into small, finite changes, we have

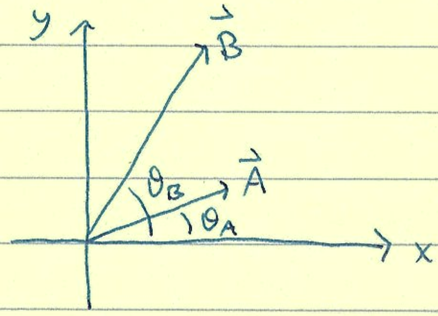
$$\Delta K = F\Delta x\cos\theta. = F v \cos\theta \, \Delta t.$$

Note that if the object has speed v, then it travels a distance $\Delta x = v\Delta t$ in time Δt. The quantity $Fv\cos\theta$ picks out only the component of force in the direction of motion.

It turns out "$Fv\cos\theta$" can be nicely encoded in a vector operation called a dot product. Consider two, two-dimensional vectors $\vec{A}$ and $\vec{B}$. Wlog, we can express their components as:

$$\vec{A} = A\cos\theta_A\hat{i} + A\sin\theta_A\hat{j}, \quad \vec{B} = B\cos\theta_B\hat{i} + A\sin\theta_B\hat{j}$$

We have the picture that:



Note that the angle between the two vectors is $\theta_B - \theta_A$.

Now, the dot product is defined as the sum of the products of each components of the vectors. Specifically, we have:

$$\vec{A}\cdot\vec{B} = (A\cos\theta_A)(B\cos\theta_B) + (A\sin\theta_A)(B\sin\theta_B)$$

$$= AB(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B)$$

$$= AB\cos(\theta_B - \theta_A).$$

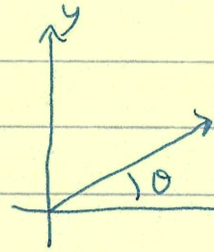
In the first line, we just multiplied x-components by x-components and summed them with y-times y-components. In the final line, we used a trigonometric identity. That is, the <u>dot product</u> is exactly what we need and picks out the shared component of two vectors.

So, we can equivalently write:

$$\frac{dK}{dt} = Fv\cos\theta = \vec{F}\cdot\vec{v}.$$

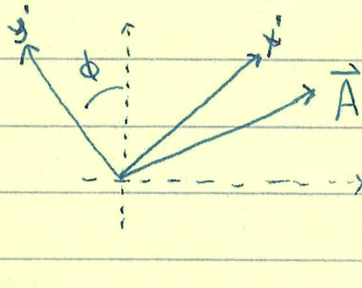
This is the work-energy theorem for motion in arbitrary dimensions.

A couple things before we end for the day. First, a <u>vector</u> as a mathematical object is defined by how it changes when undergoing rotation. For instance, $\vec{A}$ vector with some coordinate system is



$$\vec{A} = A\cos\theta\,\hat{\imath} + A\sin\theta\,\hat{\jmath}$$

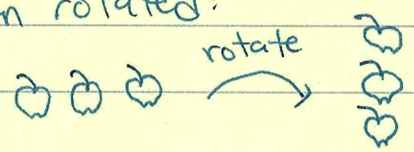
For coordinates rotated by ϕ with respect to these coordinates the vector is:



$$\vec{A} = A\cos(\theta-\phi)\,\hat{\imath} + A\sin(\theta-\phi)\,\hat{\jmath}$$

Rotation changes the direction of the vector with respect to coordinate axes, but does not change its magnitude.

A <u>scalar</u> as a mathematical object is unchanged under rotation. Three apples are still three apples when rotated:



Let's consider the rotation of the dot product. For our vectors $\vec{A}$ and $\vec{B}$ from earlier, their angles with respect to the x-axis would transform under a rotation of ϕ as:

$$\theta_A \rightarrow \theta_A - \phi \quad , \quad \theta_B \rightarrow \theta_B - \phi$$

However, their dot product transforms as:

$$\vec{A} \cdot \vec{B} = AB\cos(\theta_B - \theta_A) \rightarrow AB\cos\left[(\theta_B - \phi) - (\theta_A - \phi)\right]$$

$$= AB\cos(\theta_B - \theta_A) = \vec{A} \cdot \vec{B}.$$

That is, the dot product is rotation-invariant! The dot product takes two vectors and returns a <u>scalar</u>, just a number.

That's it for this week! Have a good weekend!