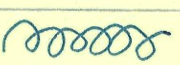
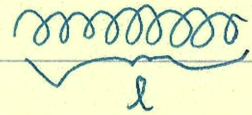
Lecture 15 Physics 101

Welcome back! Hope you had a good weekend, and please turn in homework. Again, there is an exam this Friday. A practice exam is posted on Moodle and I encourage you to review questions from conference to study, as well. There will be no conference this week; your conference leaders will have office hours then, so take advantage of it!

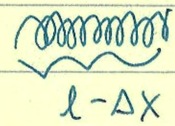
Today, we are going to introduce the approximations to end all approximations: the spring, or simple harmonic oscillator. To first approximation, almost everything in Nature is modeled as a spring in physics. In my own research, interactions of elementary particles are modeled as mediated by a spring. In fact, the calculational technique used generally in particle physics called Feynman diagrams after Richard Feynman uses a spring drawing to denote a gluon, the force carrier of the strong nuclear force:

 ← gluon

So what's the deal with springs and why are they everywhere? To answer this question, we need to figure out what force a spring can exert on an object. First, if you just encountered a spring on the street, it would likely be smoking Pall-Malls, wearing a trenchcoat; that is to say, relaxed. A relaxed spring is one for which is neither extended nor compressed; it assumes a length l when no forces act on it:

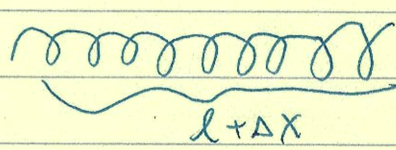


Now, let's imagine compressing the spring, from relaxed length l to $l - \Delta x$:

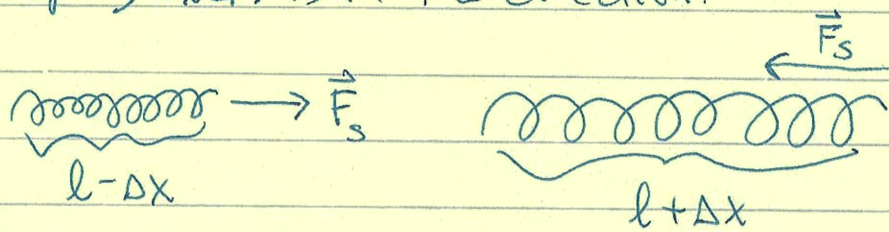


← If you've ever tried to compress a spring, what do you notice about the force you must apply as Δx increases? It gets harder and harder to compress the spring as Δx increases! This is unlike gravity or friction forces that we have dealt with thus far: both of those (so far) have been <u>constant</u> forces, independent of position. Additionally a compressed spring wants to return to the relaxed position. That is, if you compress a spring you feel a strong force pushing your hands apart.

One can do a similar thing with a spring extended to length $l + \Delta x$:



Now, to extend the spring, you have to pull your hands apart with more and more force as Δx increases. These observations suggest that the force a spring can exert on an object is monotonic with Δx. Further the spring force is a <u>restoring force</u>: the force a spring exerts acts to return ("restore") a spring to its relaxed length l. That is, ~~for the~~ the force that the spring exerts is in the direction:



These considerations motivate <u>Hooke's Law</u> for the force of an (ideal) spring:

$$\vec{F}_s = -k\Delta x\,\hat{\imath}\,, \text{ for a spring oriented along the } x\text{-axis.}$$

Here, Δx is the difference in length of the spring currently and it's relaxed length. k is called the <u>spring constant</u> and is a measure of the "stiffness" of the spring: larger k means stiffer spring (more force for same Δx). The overall negative sign indicates that this is indeed a restoring force: the direction of force opposes the direction of compression ($\Delta x < 0$) or extension ($\Delta x > 0$).

Hooke's law is named after Robert Hooke, a contemporary of Newton. Hooke often use ciphers to diguise his scientific discoveries. He first described the law that now bears his name in a Latin anagram:

ceiiinosssttuv

whose solution is

Ut tensio, sic vis

which translates to:

As the extension, so the force

Hooke also had a famous scientific rivalry with Newton, after criticizing Newton's theory of optics. As the president of the Royal Society, Newton saw to it that Hookes' scientific writings and even portraits of him were destroyed.

So, why is the spring so universal in physics? It is because of its simplicity. A general function $f(x)$, can, under "reasonable" assumptions be expressed in a polynomial-like form, called a Taylor series:

$$f(x) = f(0) + x \left.\frac{df}{dx}\right|_{x=0} + \frac{x^2}{2} \left.\frac{d^2f}{dx^2}\right|_{x=0} + \dots$$

If x is small, near 0, then higher powers of x are small and can often be neglected. If x is small enough that we can neglect x^2 and higher terms and $f(0)=0$, the function approximates to a line:

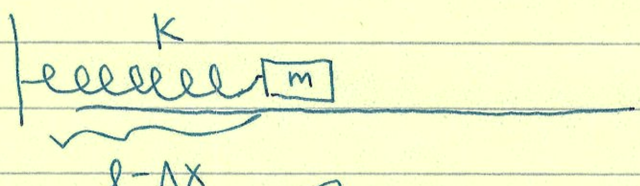
$$f(x) \cong x \left.\frac{df}{dx}\right|_{x=0}.$$

Hooke's law is simply the Taylor expansion of a force that depends on relative displacement Δx, expand to linear order in Δx:

$$F(\Delta x) \cong \Delta x \left.\frac{dF}{d\Delta x}\right|_{\Delta x=0}$$

Hooke's law says that $\frac{dF}{d\Delta x} = -k$, the spring constant. Springs, or Hooke's law is so ubiquitous in physics because to first approximation (as defined by the Taylor series) almost <u>every</u> force anywhere is a spring force.

We recently learned about work, so let's see if we can figure out how much work a spring would do on an object. Let's say we have compressed a spring a distance Δx and at the end of the compressed spring we place a block of mass m. The block is on a frictionless surface and the other end of the spring is connected to a wall, to prevent it from flying away.



The spring constant is k.

If we then let go of the block, how much work will the spring do on the block? First, the force that the spring exerts on the mass is:

$\vec{F}_s = k\Delta x\,\hat{i}$, where we assume k and Δx are both positive. As the spring expands, pushing the block, it only acts to push the block up to the point it reaches its relaxed length, and then no longer pushes the block. Thus, the spring only exerts a force over the compressed distance, Δx. The spring force is exerted in the direction of the block's motion, so the work done is:

$$W = \int_0^{\Delta x} F_s\, d\Delta x' = \int_0^{\Delta x} k\Delta x'\, d\Delta x' = \frac{1}{2}k(\Delta x')^2\Big|_0^{\Delta x}$$

$$= \frac{1}{2}k\Delta x^2$$

This is an expression we will come back to over and over: the energy that a spring imparts on an object when compressed (expanded) by a distance Δx is $\frac{1}{2}k\Delta x^2$.

By the work energy theorem, this is equal to the kinetic energy that the block gains:

$$\Delta K = W = \frac{1}{2}k\Delta x^2 = \frac{1}{2}mv^2.$$

So, the speed of the block after this spring-ing is

$$v = \Delta x\sqrt{\frac{k}{m}}.$$

It's useful to see how this also follows from simple dimensional analysis. The units of the spring constant are:

$$[k] = \left[\frac{F}{x}\right] = \left[\frac{ma}{x}\right] = M T^{-2}$$

For a spring with spring constant k, Δx compressed distance, and pushing a mass m, the mass's velocity after the spring push can be written as:

$$v = k^a \Delta x^b m^c$$

Expanding it out in basic units, note that

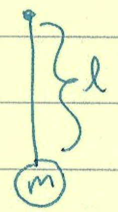
$$[k^a \Delta x^b m^c] = M^a T^{-2a} L^b M^c = M^{a+c} L^b T^{-2a}$$

For this to equal a speed, $[v] = L T^{-1}$, we must have that:

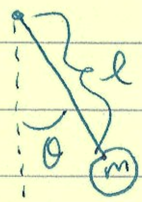
$a = -c$, $b = 1$, $a = \frac{1}{2}$ so then $c = -\frac{1}{2}$. That is, we find that

$v = k^{1/2} \Delta x \, m^{-1/2}$, exactly what the work-energy theorem predicts!

In the remaining time, I want to introduce the pendulum, the system of a suspended, swinging mass. Say a mass m is tied to the end of a string of length l like so:

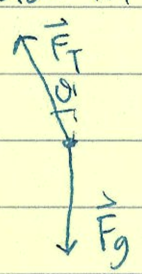


Now, we pull back the mass an angle θ from the vertical like

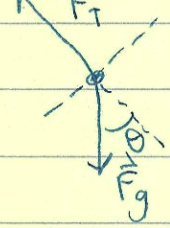


What is the net force on the mass? right after we let go?

As always, let's draw our free-body diagram. The only forces acting on the mass are gravity, and the tension in the string:



As the mass swings back and forth, it travels in a fixed radius trajectory, along the arc of a circle. Therefore, there must be <u>some</u> centripetal acceleration keeping the mass in this arc. This suggests that to analyze these ~~masses~~ forces, we should align an axis with the string, and the other perpendicular to it. So, we have:



The net force in the direction of the string would be:

$F_T - mg\cos\theta = m\frac{v^2}{l}$, as this force is centripetal, and responsible for movement along a circular arc. The force perpendicular to this, tangent to the arc, is:

$-mg\sin\theta = ma_T$, where a_T is the tangential acceleration. This is equivalently

$$-mg\sin\theta = ma_T = m\frac{dv}{dt} = ml\frac{d\omega}{dt},$$ where we

note that $v = l\omega$, and ω is the angular velocity,

$\omega = \frac{d\theta}{dt}$. That is, Newton's law tangent to the arc reduces to:

$$-g\sin\theta = l\frac{d\omega}{dt} = l\frac{d}{dt}\frac{d\theta}{dt} = l\frac{d^2\theta}{dt^2}$$

or that $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$.

Written as it is now, this is a bit hard to parse as that $\sin\theta$ factor is scary. However if θ is a small angle, we can Taylor expand $\sin\theta$ as:

$$\sin\theta = \sin 0 + \theta \left.\frac{d\sin\theta}{d\theta}\right|_{\theta=0} + \ldots = \theta + \ldots$$

$\sin 0 = 0$ and $\frac{d\sin\theta}{d\theta} = \cos\theta$, and $\cos 0 = 1$ so $\sin\theta \cong \theta$.

So, we find that:

$$\boxed{\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta}$$

The angular acceleration of the pendulum, $\frac{d^2\theta}{dt^2}$, is linearly proportional to θ, and further is a restoring force (negative sign). Compare this to Newton's second law for a spring:

$$-k\Delta x = ma = m\frac{d^2\Delta x}{dt^2} \quad \text{or that} \quad \frac{d^2\Delta x}{dt^2} = -\frac{k}{m}\Delta x.$$

Apparently, a pendulum is just a spring!

Let's see how this works! Demo time!

See you Wednesday!