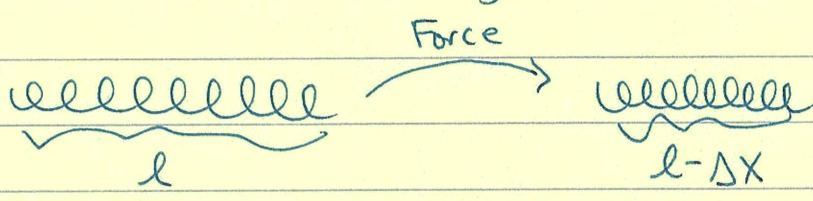
Lecture 16 Physics 101

Last lecture before the exam. Woohoo! I hope you are taking advantage of the embiggened office hours this week. On Friday, please come with your brain, a writing utensil, perhaps a calculator (not needed), and an awesome attitude! :)

Also, the homework assigned today will be due next Monday, October 14. No homework is due on the date of the exam.

Over the past week, we have introduced the concept of energy, how it can be conserved, the work-energy theorem, and the work of a spring. Today, we will discuss the important concept of potential energy, energy that is stored in an object or system that can be utilized later to perform some task (remember our definition of energy?). A canonical example of potential energy is that stored in a compressed spring. Think about it: initially the spring is relaxed. To compress the spring an amount Δx, you have to exert a force on the spring over a distance Δx. Therefore you did work on the spring!



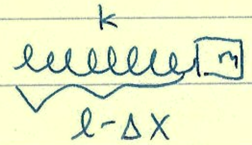
However, the work you did to compress the spring didn't change the spring's kinetic energy: the spring is still at rest. However, you clearly exerted energy (that is, did work), and that energy can't have vanished into the aether. So, where did it go? The work you peformed on the spring transferred to the spring's <u>potential</u> to do work on another object. As discussed last lecture, when compressed by an amount Δx, a spring will do work on a mass at the end of the spring of:

$$W = \frac{1}{2} k \Delta x^2, \quad \text{where } k \text{ is the spring constant.}$$

The work you did on the spring to compress~~ed~~ it is then stored in the potential ability for the spring to do an amount of work equal to $\frac{1}{2} k \Delta x^2$ on an object. More compactly, we say that a spring compressed by an amount Δx has potential energy U equal to

$$U = \frac{1}{2} k \Delta x^2.$$

If we put a box of mass m at the end of a spring which had potential energy of $\frac{1}{2} k \Delta x^2$, then this would be transferred into kinetic energy of the box once the spring returned to its relaxed length. If we thought of our universe as solely consisting of the block/spring system with no friction whatsoever, then the only types of energy allowed are the potential energy of the spring and the kinetic energy of the block. That is, for the system:



if it is isolated/closed then its total energy must be conserved as there is no way for energy to be lost or gained. Therefore, the sum of kinetic and potential energy of this system is constant in time:

$$U + K = \frac{1}{2} k \Delta x^2 + \frac{1}{2} m v^2 = \text{constant} \equiv E,$$

where E is the total energy. Demanding that total energy be constant in time is typically vastly simpler for analyzing a problem than using Newton's second law, even though they lead to equivalent results.

Another thing to note about this spring potential energy is its simple relationship to Hooke's law. Let's take a derivative of $U = \frac{1}{2} k \Delta x^2$ with respect to Δx:

$$\frac{dU}{d\Delta x} = \frac{1}{2} k \frac{d\Delta x^2}{d\Delta x} = \frac{1}{2} k \cdot 2 \Delta x = k \Delta x = -F_{spring},$$

by Hooke's law. That is, we note that

$$F_{spring} = -k\Delta x = -\frac{dU}{d\Delta x}$$

Forces for which they are related to a potential energy by this negative derivative are called conservative forces. The name doesn't connote US political parties, but rather that the work done by such a force is exclusively from potential energy decrease (hence the "−" sign). Conservative forces, well, conserve energy all on their own.

We'll study another conservative force in a second, but it's important to note that not all forces are conservative. Perhaps the most familiar example is the force of friction. As we have discussed, friction can do work on an object to change its kinetic energy, but in doing so, that kinetic energy is transformed into many different forms of energy (heat, sound, etc.). The work friction does on an object does not exclusively turn that kinetic energy into potential energy, like with an (ideal) spring. We therefore say that friction is a non-conservative force. For friction, it is not possible to express it as a derivative of a potential energy.

Enough about non-conservative forces for now; let's get back to conservative forces and perhaps the most familiar force of all: gravity. First, let's argue that gravity is indeed conservative. Well, where does all of the work that gravity does on an object go to? Into changing the kinetic energy of the object! We can imagine a world without air, friction, etc., and gravity would still be doing its thing, keeping thrown balls in parabolic trajectories. Therefore, we can use the conservative force formula to determine the potential energy of gravity. Considering the force of gravity exclusively in one dimension (up and down) the force of gravity is:

$$F_g = -mg,$$

when acting on an object of mass m. The corresponding potential energy is found from integrating from an initial height h_1 to a final height h_2:

$$U = -\int_{h_1}^{h_2} F_g \, dx = mg \int_{h_1}^{h_2} dx = mg(h_2 - h_1).$$

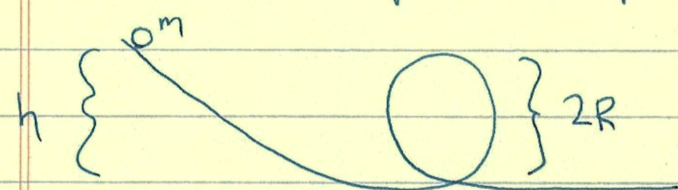
That is, the gravitational potential energy is linear in the height difference from the current position to a reference position.

Unlike the potential energy of a spring, gravitational potential energy can be positive or negative in sign. This may seem weird, but all that matters are potential energy differences for determining how gravity affects an object's kinetic energy. That is, the statement of conservation of energy for an object of mass m exclusively acted on by gravity is:

$$E = mgh + \frac{1}{2}mv^2 = \text{constant},$$

where h is the height above a vertical origin point.

This is all still a bit abstract, so let's consider a concrete, real system in which we can test this whole "conservation of energy" stuff. What I have here is a loop-the-loop set up:



A ball travels down the ramp and then through the loop, ultimately traveling out, off to the right. We would like to predict the height h such that the ball stays on the track throughout traveling through the loop. The radius of the loop is R. We'll test this out once we have a prediction.

While we won't solve it this way, let's imagine that we attempt to solve this with Newton's second law directly. Where do we even start? We would need free body diagrams to determing the speed of the ball at the end of the ramp, which is easy enough. However, imagine the FBDs for analyzing when the ball is in the loop! The direction and magnitude of the forces on the ball constantly change, so this would be a nightmare to analyze.

Luckily, we have energy on our side. We simply need to evaluate the initial energy and the final energy, equate them, and we can solve for height. So let's do this!

Initially, the ball is released from rest a height h above the ground. Initial kinetic energy is therefore 0, so the total initial energy is:

$$E_i = mgh.$$

Now, if the ball is suppose to reach the top of the loop and remain on the track two things must happen. First, the ball has to actually have enough energy to even reach a height of $2R$ (the top of the loop), but further, if the ball is still on the track, then it is traveling in a circle. As such there must be a centripetal acceleration acting on the ball at the top of the loop. Gravitational force is always there so, at least, the centripetal acceleration is g. If the centripetal acceleration is g, then the ball has a minimal speed at the top of the loop:

$$a_{cent} = \frac{V^2}{R} = g, \text{ or that } V_{min}^2 = Rg.$$

This correspondingly implies that there is a minimal kinetic energy that the ball has at the top of the loop. This kinetic energy is

$$K_{min} = \frac{1}{2} m V_{min}^2 = \frac{1}{2} Rg.$$

The ball also has gravitational potential energy, as it has a nonzero height above the ground. This potential energy is:

$$U = mg(2R),$$

as the ball is a height 2R from the ground at the top of the loop.

So, the total energy of the ball at the top of the loop must be at least

$$E_{top} = K_{min} + U = \frac{1}{2} Rg + mg(2R) = \frac{5}{2} mgR.$$

By conservation of energy, this has to equal the initial potential energy of the ball a height h above the ground. This therefore enables us to solve for this (minimum) height h as:

$$E_i = E_{top} \Rightarrow mgh = \frac{5}{2} mgR \text{ or that } h = \frac{5}{2} R.$$

Let's try this out! Let's see if the ball indeed stays in the lopp if the initial height is at least $5/2\, R$.

That's it for today; see you on Friday for the exam!