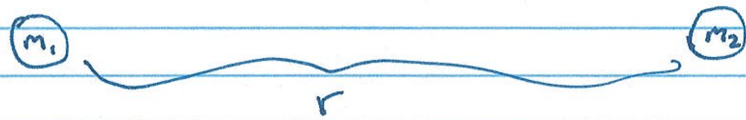
Physics 101 Lecture 18

Welcome to the last contentful lecture before fall break! Please turn in homework and such. (Not sure what "such" means.)

Today we are going to create a theory of gravity that generalizes our simple discussion of a uniform force, universally pulling objects toward the ground. This theory that we will create will subsume our constant accelerating gravity, and explicitly predict when that assumption breaks down.

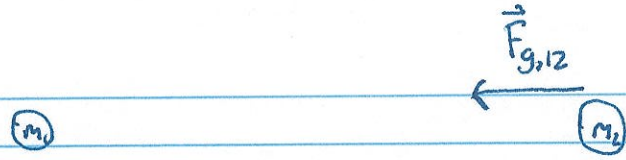
To construct this theory of gravity, we will imagine placing two objects of mass m_1 and m_2 a distance r apart:



We would like to determine the force of gravity from mass 1 on mass 2, $\vec{F}_{g,12}$. Note also that we assume that these two masses are balls, but we will actually work in the approximation that they are points, and have no spatial extent. If point masses make you uncomfortable, then equivalently, we work in the limit in which the distance between the objects r is much larger than either of their individual radii:

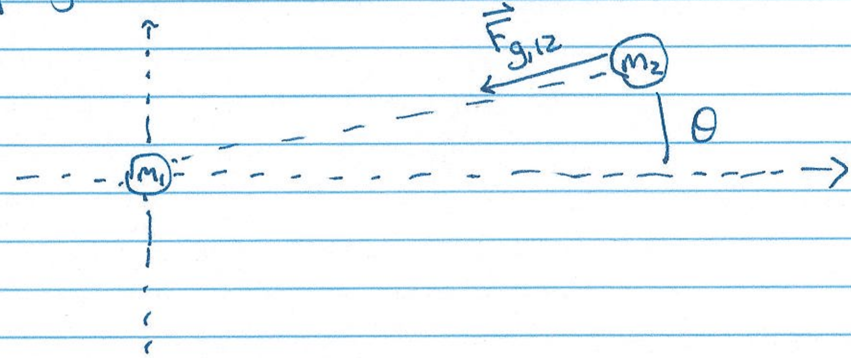
$$r \gg r_1, r_2$$

The gravitational force on 2 by 1 is a vector, so we need to determine both its magnitude and direction. Let's start with the direction. Gravity is a universally attractive force, meaning that two masses are always attracted to one another via gravity. Specifically, in the case at hand, the direction of the force on mass m_2 points ~~two~~ toward m_1:



Now, with this particular alignment, $\vec{F}_{g,12} = -F_{g,12}\,\hat{\imath}$,

where $F_{g,12}$ is the magnitude of $\vec{F}_{g,12}$: $F_{g,12} = |\vec{F}_{g,12}|$. However, we can orient our axes to describe the positions of the two masses however we want. A convenient orientation is with m_1 at the origin and m_2 an angle θ above the horizontal, when projected on two-dimensions:



With this orientation, the gravitational force is:

$$\vec{F}_{g,12} = -F_{g,12}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right)$$

Regardless of θ, the gravitational force vector points ~~two~~ toward the origin, along a line that emanates from the origin. Such lines are nothing more than radial lines (they "radiate" from the origin) and we denote the vector with unit length that points along a radial line as $\hat{r}$:

$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

Another way to say this is that there is a rotational symmetry about mass m_1: rotating mass m_2 any angle about m_1 leaves the magnitude of gravitational force the same, and just rotates its direction to always point toward m_1. Thus, the gravitational force is:

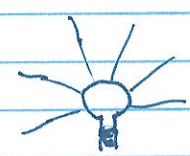
$$\vec{F}_{g,12} = -F_{g,12}\,\hat{r}.$$

Okay, we have the direction; what about magnitude? In general, as they are given quantities, the gravitational force could depend on distance r and the masses m_1 and m_2:

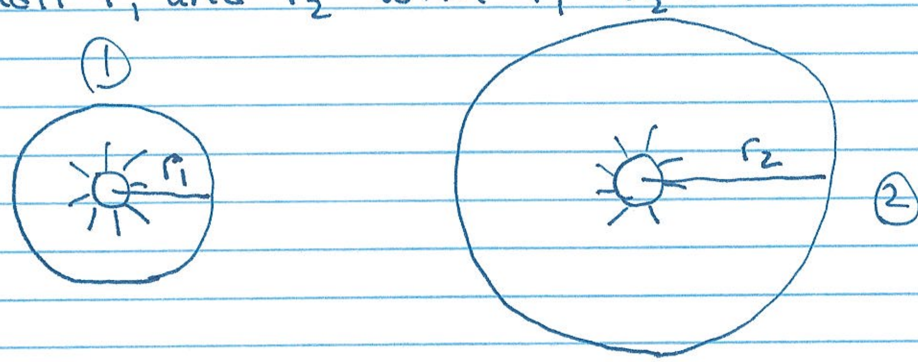
$$F_{g,12} = F_{g,12}(r, m_1, m_2)$$

Let's focus on the distance dependence first.

We live in three spatial dimensions, which may be obvious, but is extremely important for determining the dependence on r. We will work by analogy here; first imagining a light-bulb hanging out in space:



This bulb emits light in all directions uniformly. Let's imagine putting the bulb inside a sphere of radii r_1 and r_2 with $r_1 < r_2$:



How does the total amount of light that is captured by the two spheres compare?

a) Sphere 1 more light   b) Sphere 2 more light

c) same amount

The bulbs in both cases are identical, outputting the same amount of light and both spheres capture all of the light from the bulb. Therefore, they capture the same amount of light.

However, imagine that you are sitting on the interior surface of the spheres. Which case would the bulb appear brighter?

a) Sphere 1 is brighter b) Sphere 2 is brighter

c) same brightness

Now, your eye is not like the sphere; it does not capture all of the light emitted by the bulb; it only captures the light that hits a very small region. The amount of light that hits a given small region of the sphere is controlled by the total amount of light from the bulb divided by the surface area of the sphere: the light per unit area. The surface area of a sphere is

$A = 4\pi r^2$, where r is its radius.

Thus, the light per unit area for the two spheres is

$\frac{L}{A_1} = \frac{L}{4\pi r_1^2}$, $\frac{L}{A_2} = \frac{L}{4\pi r_2^2}$, where L is a measure of the total amount of light from the bulb.

Because $r_2 > r_1$, the light per unit area for sphere 2 is smaller than for sphere 1, so you perceive the light in that case as dimmer. Note that the perceived brightness follows an "inverse square law": if the radius of the sphere doubles, the perceived brightness decreases by a factor of 4.

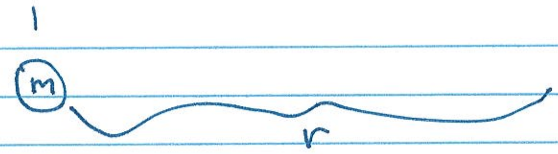
Now, let's take this observation to understand gravity. Our universe with mass 1 and mass 2 is still three-dimensional, and as we have emphasized throughout this class, forces always have some agent. Let's hypothesize that this agent for gravity is similar to the light from the bulb. That is, the total effect of gravity from mass m_1 is constant and only depends on properties of mass m_1. However, the density of the agent that exerts gravitational force on mass m_2, like the light, would decrease like $1/r^2$. Thus, with this hypothesis, the force of gravity would also follow an inverse square law:

$$F_{g,12}(r, m_1, m_2) = \frac{1}{r^2} M(m_1, m_2),$$

where $M(m_1, m_2)$ is a function purely of the masses m_1, m_2. Again, in science we don't need to answer "why?" for every question to make progress. We can hypothesize and test our hypothesis and learn something about the universe. We don't need to answer the question of what the gravitational force agent is to test our theory of gravity.

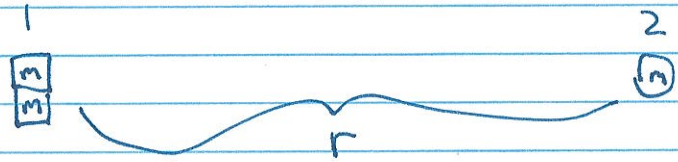
Now, let's figure out the mass dependence of the gravitational force. We use the equivalence principle, so gravitational mass and inertial mass are equivalent and just "mass". Mass is a measure of how much "stuff" an object is constructed from (I don't know what "stuff" is, however.) So, we will answer the question of how the amount of stuff affects gravity.

To proceed, we will additionally assume that the effects of gravity are linear: that is, the net gravitational force on a object by two objects is simply the sum of individual forces. We will use this in a second. First, let's imagine that $m_2 = m_1 = m$, some basic unit of force. Then, from what we have constructed above, the gravitational force in this case is:



$$\Rightarrow F_{g,12} = \frac{M(m,m)}{r^2}$$

Now, let's imagine making $m_1 = 2m$, keeping $m_2 = m$. We can visualize this as:

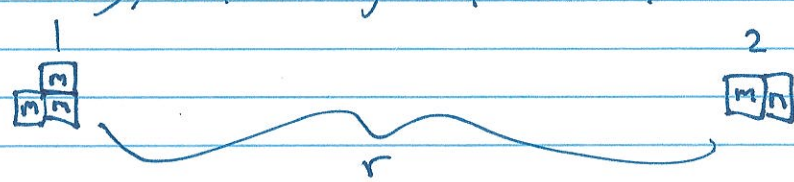


Where we imagine putting two blocks of mass m each at the location 1. By linearity of gravity, to find the force on mass 2, we can sum together the gravitational forces of the blocks individually:

$$F_{g,12} = \frac{M(m,m)}{r^2} + \frac{M(m,m)}{r^2} = \frac{2M(m,m)}{r^2}$$

Note that we don't have to worry about vector addition because we assume that the blocks are at the same point.

Continuing, let's imagine that $m_1 = 3m$ and $m_2 = 2m$:



Using linearity of gravity, what is the force on m_2?

a) $F_{g,12} = \frac{3M(m,m)}{r^2}$ b) $F_{g,12} = \frac{6M(m,m)}{r^2}$

c) $F_{g,12} = \frac{2M(m,m)}{r^2}$

If $m_1 = 3m$ and $m_2 = 2m$, then there are a total of 6 pairs of masses in which one comes from m_1

and the other in the pair comes from m_2. 6 is simply the product of the relative masses of m_1 and m_2:

$$\frac{m_1}{m}\frac{m_2}{m} = \frac{3m}{m}\frac{2m}{m} = 6$$

So, ~~gm~~ generalizing, if $m_1 = N_1 m$ and $m_2 = N_2 m$, where N_1 and N_2 are positive numbers, the gravitational force on mass 2 is:

$$F_{g,12} = \frac{N_1 N_2 M(m,m)}{r^2}, \text{ or we can express it as}$$

$$F_{g,12} = \frac{G_N m_1 m_2}{r^2},$$ proportional to the product of masses m_1 and m_2.

G_N is a constant of proportionality, called Newton's constant, to ensure that units are correct. With force having units of kg m/s², the units of G_N are:

$$[G_N] = \left[\frac{Fr^2}{m^2}\right] = MLT^{-2}L^2M^{-2} = M^{-1}L^3T^{-2}$$

or in SI, $[G_N] = kg^{-1}\,m^3\,s^{-2}$.

Thus, the force vector between two masses is

$$\boxed{\vec{F}_{g,12} = -\frac{G_N m_1 m_2}{r^2}\hat{r}}$$

This is called Newton's Universal Law of Gravitation.

That's it for today! Come in Friday for something fun (I hope!).