Lecture 19 Physics 101

Welcome back from break! I hope it was relaxing and restful. Now back into the fray!

While there is no homework due today, I want to remind you that there is a new homework assigned today, to turn in Wednesday (as usual).

In our last substantive lecture, Wednesday before break, we had argued for the form of the inverse-square law form of the gravitational force between two masses m_1 and m_2 separated by distance r:

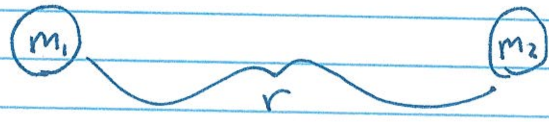
$$|\vec{F}| = \frac{G_N m_1 m_2}{r^2}$$

G_N is called Newton's constant and has the value of $G_N = 6.67 \times 10^{-11}\ \mathrm{Kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$, and this sets the strength of the gravitational force between the two masses. If this value were larger, the force would be larger, and if it were smaller, the force would be smaller. The fact that it is of order 10^{-11} in SI units means that the strength of gravity is <u>very</u> weak. The <u>entire</u> mass of the Earth pulls you down, but you can still jump up, off ~~ea~~ Earth, using your measly legs!

Today we are going to study the gravitational force in another way, to expose the energy that it can store. This will also segue into one of the most mysterious objects in the universe.

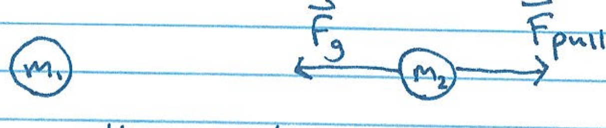
Let's attempt to address the question: How much energy would it take to blow up Earth? Now, this isn't some fatalisitic take on today's society, we want to determine the amount of energy ~~to~~ it takes to completely pull apart every rock, every atom of Earth. Earth is held together through the gravitational force of its constituents, so to pull the Earth apart, we need to do work against gravity to do this. Note that the range of the gravitational force is infinite: as long as the distance r between two massive objects is not ∞ ($r < \infty$), then their gravitational force is non-zero. So, to completely blow up the Earth, we need to pull all of its atoms apart and an infinite distance from one another. This is a really tall task, so we will simplify our picture of the Earth to analyze this.

Our model of the Earth will be the following: two masses m_1 and m_2 separated by distance r:



Okay, okay, so not very realistic. However, if r is about the radius of Earth and m_1 and m_2 are about half the mass of Earth, then by analyzing this system, we will be able to determine how much energy it would take to break Earth in two pieces. If we did that, I might claim victory. :)

Okay, to separate the masses, we need to do work against the force of gravity. What we will imagine doing is pulling mass m_2 from a separation of r with mass m_1 to a separation of ∞. We will just pull mass m_2 to the right to do this, which is opposite to the direction of gravitational force:



If we pull such that m_2 travels at a constant velocity, then $|\vec{F}_{pull}| = |\vec{F}_g|$, and we can determine how much work we would need to do to accomplish this. We do work from a distance r to a distance ∞, pulling in the direction of motion with a force equal in magnitude to gravitational force. That is, the work we need to do to separate the masses is:

$$W = \int_r^\infty \vec{F} \cdot d\vec{r}\,' = \int_r^\infty |\vec{F}_g|\, dr' = \int_r^\infty \frac{G_N m_1 m_2}{r'^2}\, dr'$$

$$= -\frac{G_N m_1 m_2}{r'}\Big|_r^\infty = \frac{G_N m_1 m_2}{r}$$

To do the integral of dx/x^2, note that the derivative of $1/x$ is:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = -x^{-2}$$

Therefore, the <u>anti</u>-derivative of x^{-2} is $-x^{-1}$. The work we had to do to separate the masses is still proportional to the product of their masses, but now only inversely proportional to their initial separation. The closer they are initially, the harder it is (the more work we have to do) to separate them.

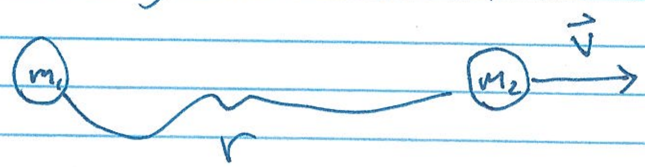
Now, gravity is a conservative force, as we discussed, so if we did this much work, then the <u>opposite</u> of this was initially stored as <u>potential energy</u>. That is, the gravitational potential energy of two masses m_1 and m_2 separated by distance r is

$$U = -\frac{G_N m_1 m_2}{r}$$

Note the $-$ sign: it takes energy from us to separate the masses. With this gravitational potential energy, we can do everything we usually do with energy. And, remember, energy is a scalar (it has no direction) so it's easy to find total energies of multiple masses interacting gravitationally: we simply sum them up.

A related question to blowing up Earth is the following. How fast would you have to throw a ball upward, away from Earth such that the ball ended up traveling an infinite distance from Earth? Within the context of our old model of gravity as a constant force, this was impossible, because constant force means eternally non-zero acceleration, so any finite velocity would eventually stop and reverse. For inverse-square gravity, we can travel fast enough to get out of the gravitational pull of Earth. We say we have "escaped Earth's gravity" and the initial velocity needed to do this is called the "escape velocity".

To determine the escape velocity, we are going back to our model of two masses m_1 and m_2. Now, however, we are going to give mass m_2 an initial velocity $\vec{v}$ pointed away from mass m_1:



How large must $|\vec{v}|$ be for m_2 to escape m_1's gravity? We can solve this with conservation of energy. Initially the masses have a gravitational potential energy of

$$U_i = -\frac{G_N m_1 m_2}{r}$$

and mass m_2 has kinetic energy $K_i = \frac{1}{2} m_2 v^2$. The total initial energy is then

$$E_{tot} = U_i + K_i = -\frac{G_N m_1 m_2}{r} + \frac{1}{2} m_2 v^2$$

Now, when m_2 has escaped m_1, it is infinity far away, so there is <u>no</u> gravitational potential energy, $U_f = 0$. Further, if mass m_2 <u>just</u> makes it out there, its final kinetic energy is 0 (no velocity). That is, the final total energy is 0: $E_{tot} = 0$. Setting initial and final energies equal, we have

$$-\frac{G_N m_1 m_2}{r} + \frac{1}{2} m_2 v^2 = 0$$

or, solving for v, we find: $v^2 = \frac{2 G_N m_1}{r}$

This value of the speed, $\sqrt{\frac{2G_N m_1}{r}}$, is called the <u>escape velocity</u>:

$$V_{esc} = \sqrt{\frac{2G_N m}{r}} \text{ for a mass } m.$$

Setting $m = M_{Earth}$ and $r = R_{Earth}$, the escape velocity from the surface of Earth is

$$V_{esc} = \sqrt{\frac{2G_N M_{Earth}}{R_{Earth}}} \cong \sqrt{\frac{2 \cdot 6.67 \cdot 10^{-11} \cdot 6 \cdot 10^{24}}{6 \cdot 10^{6}}} \frac{m}{s} \approx$$

$$\cong 12 \text{ km/s} \cong 25{,}000 \text{ mph}$$

Note that escape velocity depends on the initial distance from the gravitating object. What we derived above the the escape velocity from Earth's surface. If you are farther away from Earth when you start, the the velocity you need to escape Earth's gravity is correspondingly less. For example the probes Voyager 1 and 2 were launched from Earth in the 1970s with speeds much less than what would be needed to escape the gravitational force of the sun, from a radius of the orbit of Earth. As they traveled through the solar system, they were able to get energy kicks from orbits around Jupiter, which pushed their velocities past the escape velocity of the sun, at a radius of Jupiter's orbit. Voyagers 1 and 2 are just two of only five artificial objects that have attained Solar escape velocity and have left the solar system.

In the time that remains, let's throw this escape velocity idea on its head. While not a topic for this class, you might know that the speed of light in vacuum is an ultimate, ~~uni~~ universal speed limit. Nothing can travel faster than light. The speed of light is typically denoted as c and in SI units is

$$c = 3 \times 10^8 \, m/s$$

Imagine that there was a massive object whose escape velocity was $c = v_{esc}$. This would mean that not even light, travelling as fast as possible, could ever escape the gravitational pull of the object. I should also say that the manipulations I am going to do now are questionable, but give the right answer, so we will use them to provide insight into properties of such a massive body. Let's say this massive body has mass M and radius R, and we assume that its escape velocity from its surface (~~at~~ radius R) is c. That is,

$$c = \sqrt{\frac{2G_N M}{R}}$$

We can solve this instead for escape velocity c, for the radius R:

$$R = \frac{2G_N M}{c^2}$$

The interpretation of this distance is the following. If a mass M is entirely contained within a sphere of radius

$R = \frac{2G_N M}{c^2}$, then the escape velocity from the surface of that sphere is the speed of light, c. As no even light can escape this mass, it is called a "black hole".

This point should be emphasized. There is no black hole at the center of the Earth, because all of Earth's mass is not concentrated there. This radius for a given mass M is called its "Schwarzschild Radius" after Karl Schwarzschild, a German physicist who first derived it, literally in the foxholes of World War I. For the Earth's mass of 6×10^{24} Kg, its Schwarzschild radius would be:

$$R_{Sch} = \frac{2 G_N M_{Earth}}{c^2} \cong 9 \text{ mm}.$$

That is, if all of Earth's mass were contained within a sphere of radius 9mm, then it would form a black hole. You could hold it in your hand, but the gravitational force would be so strong you would quickly be sucked in to it!

If we have a bit more time we will talk about the observation of M87's Black Hole.