Lecture 20 Physics 101

Please turn in homework! Welcome back to Physics for more fun!

We are going to leave gravitation for a bit, and introduce another conservation law. Let's start with Newton's second law for a single object or particle of mass m:

$$\vec{F}_{net} = m\vec{a}$$

Now, as a single entity, we are also going to imagine that the mass cannot change: that is, the object can't gain or lose mass. Parts of it can't fall off and nothing can stick to it. With this assumption, we can re-express Newton's second law as:

$$\vec{F}_{net} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right) = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

we call $\vec{p}$ the <u>momentum</u> of the object, but for now it is simply a placeholder name for the quantity $m\vec{v} = \vec{p}$.

Written in this form, however, Newton's second law has a nice interpretation: if $\vec{F}_{net} = 0$, then the time derivative of momentum $\vec{p}$ is 0, or that momentum does not change in time. That is, if $\vec{F}_{net} = 0$, then the particle's momentum is <u>conserved</u>. While this sounds profound, at this stage it is nothing more than the statement that if there are no forces, then there is no acceleration, so the particle's velocity is unchanged in time.

Further, just like we did when introducing work and energy, we can anti-differentiate Newton's second law to determine the change in momentum from a force that acts over <u>time</u>. Recall that work is from a force that acts over distance, while we call <u>impulse</u> the effect of force acting over time. We can integrate Newton's law over time:

$$\int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \int_{t_1}^{t_2} \vec{F}_{net}\, dt = \Delta\vec{p},$$

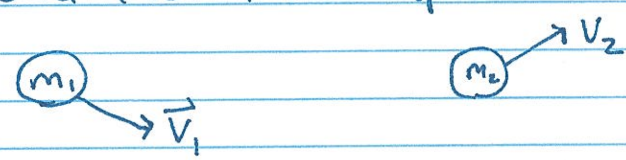
where $\Delta\vec{p}$ is the change in the momentum from time t_1 to time t_2:

$$\Delta\vec{p} = \vec{p}(t_2) - \vec{p}(t_1)$$

We might call this the "impulse-momentum" theorem, in analogy to the work-energy theorem, but that is not typically used.

One final point before moving on: as we are currently considering a localized, isolated object, we imagine that it effectively has no extent. On a free-body diagram it is but a point, so all of its mass m is localized at its position $\vec{x}$. The point at which mass is or can be imagined to be localized is called the "center-of-mass" of the system. We'll need this in a second...

Okay, enough of one particle, let's imagine we have a system of two particles of mass m_1 and m_2 and we want to determine that systems dynamics with Newton's second law. The set up is:



and we give the masses some velocities $\vec{v}_1$ and $\vec{v}_2$. We can write down Newton's second law for each mass individually:

$$\vec{F}_{net,1} = \frac{d\vec{p}_1}{dt}, \quad \vec{F}_{net,2} = \frac{d\vec{p}_2}{dt}$$

Note that the net forces on mass 1 may include forces exerted by mass 2, and vice-versa, if, for example, gravity is a relevant force. However, by Newton's third law, every force that 2 exerts on 1 has an equal and opposite partner of 1 exerts on 2. So, if we consider the <u>total</u> system of masses 1 and 2 together, then the only forces that affect the system are external to the two masses:

$$\vec{F}_{ext} = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = \vec{F}_{net,1} + \vec{F}_{net,2}.$$

Apparently, if there are ~~not~~ external forces, then the <u>sum</u> of the particle momenta is conserved. Forces only exerted between the particles do <u>not</u> affect the total momentum, by Newton's third law.

Let's keep going and attempt to interpret what the sum of momentum is. Again, assuming for simplicity that the individual particle masses are constant, we have:

$$\vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 = \frac{d}{dt}\left(m_1\vec{x}_1 + m_2\vec{x}_2\right)$$

The term in parentheses is the mass-weighted position of the particles. Apparently, if there are no external forces, then there is no acceleration of this mass-weighted position. Let's go a bit farther, multiplying and dividing by the <u>total</u> mass:

$$\vec{p}_1 + \vec{p}_2 = (m_1 + m_2)\frac{d}{dt}\left(\frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}\right)$$

Now the quantity on the right is called the <u>center-of-mass</u> and is the location at which <u>all</u> of the mass of the system can be imagined to be, for the purpose of where forces act. We denote this as

$$\vec{X}_{cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$

Note that, if $m_1 \to 0$, then all of the mass is confined to be at mass 2, so $\vec{x}_{cm} = \vec{x}_2$ (and similar if $m_2 = 0$). So, another way to express the sum of momentum of the two particles is:

$$\vec{p}_1 + \vec{p}_2 = (m_1 + m_2)\frac{d\vec{x}_{cm}}{dt} = (m_1 + m_2)\vec{v}_{cm},$$

where $\vec{v}_{cm}$ is the velocity of the center-of-mass. Then, Newton's second law can be re-expressed as:

$$\vec{F}_{ext} = (m_1 + m_2)\vec{a}_{cm},$$

where $\vec{a}_{cm}$ is the acceleration of the center-of-mass. That is, if there are no external forces, then the center-of-mass does not accelerate.

For a system of many particles, the story is the same; we just need to sum over all of their individual momenta. In that case, Newton's second law for a system of n particles is:

$$\vec{F}_{ext} = \vec{a}_{cm}\sum_{i=1}^{n} m_i = \frac{d}{dt}\left(\sum_{i=1}^{n}\vec{p}_i\right)$$

where $\sum_{i=1}^{n} m_i = m_1 + m_2 + \ldots + m_n$, $\sum_{i=1}^{n}\vec{p}_i = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n$.

We believe that there is nothing external to our universe, so necessarily there are no net external forces on our universe:

$$\vec{F}_{net,universe} = 0.$$

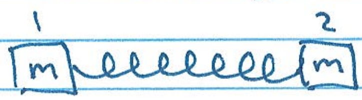
This then implies that the sum of the momenta of <u>all</u> particles in the universe is conserved, unchanging in time.

Now, to say that the center-of-mass does not accelerate does not mean that it cannot move when there are no external forces. For example, let's consider a couple different configurations of particles. First, let's assume that masses m_1 and m_2 are identical and equal to m. What is the velocity of the center-of-mass if $\vec{x}_1(t) = -\vec{x}_2(t)$, for all t?

Well, we find this by simply plugging this into the expression for center of mass:

$$\vec{X}_{cm} = \frac{m\vec{x}_1 + m\vec{x}_2}{m+m} = \frac{1}{2}\left(\vec{x}_1 + (-\vec{x}_1)\right) = 0$$

The masses can move, but the center-of-mass does not. An example of such a system would be two masses connected by a (massless) spring:



We can compress the spring and the masses will just oscillate back and forth, with $\vec{x}_1(t) = -\vec{x}_2(t)$, but won't be drifting anywhere in space. Correspondingly, if the center-of-mass doesn't move, then the net momentum is 0.

While this configuration had no net motion of the center-of-mass, we had discussed some time ago that there is no such thing as absolute velocity, so we could imagine this mass-spring system moving by at constant velocity, and there still be no net forces. However, now in this case there would be a non-zero momentum of the system because the center-of-mass moves. This observation connects to the Noether's theorem interpretation of momentum.

If there are no external forces on our system, then the net momentum of the system is conserved. By Noether's theorem, if momentum is conserved, then there should be a corresponding symmetry under which the system of objects/particles is unchanged. As discussed, The net momentum of a system is intimately related to the motion of its center-of-mass, $\vec{x}_{cm}$. The center-of-mass is some position in space and to move the center-of-mass requires spatial translation. For example, if the center-of-mass is initially at $\vec{x}_{cm} = (1m)\hat{i}$ and we want to move it to $\vec{x}_{cm} = (2m)\hat{i}$, then we need to translate one meter to the right.

We know how to do this translation: we simply give the system a non-zero momentum and the center-of-mass will move. The statement of conservation of momentum means that, by Noether's theorem, we can move the center-of-mass anywhere, and the physics of the system is unchanged. That is, if our system is invariant (=unchanged) by any spatial translation, then total momentum is conserved. This is Noether's theorem for spatial translations:

Noether's Theorem	
Symmetry	Conservation Law
Spatial Translations	Momentum, $\vec{p}$

Momentum conservation, like energy conservation, is typically much easier and more useful to directly use than Newton's second law to analyze the dynamics of a system. When only internal forces are relevant; i.e., forces between objects in the system and no forces from the outside, momentum is conserved, and this is typically what is relevant for collisions analysis. Indeed, in my research which studies the collisions of protons at high energies, momentum conservation is extremely important for constraining the physics that may have been produced. We'll discuss more about collisions on Friday.

For the last few minutes today, I just want to briefly connect what we have learned to the broader and more general context. We have argued that if energy is conserved, the laws of physics are independent of time. Correspondingly, if momentum is conserved then the laws of physics is independent of spatial position. Independence of time or position means that the ~~t~~ derivative of the laws of physics with respect to these quantities is zero. Let's denote the laws of physics compactly as S. Conservation of energy means that

$$\frac{d}{dt}S=0$$

and conservation of the momentum vector means that

$$\frac{d}{dx}S=\frac{d}{dy}S=\frac{d}{dz}S=0$$

"Laws of physics" isn't just a simple function, ~~so~~ as S must encode the motion and interactions of all particles in the universe. As such, S is a function of all the particles' trajectories, which is itself a function of t, x, y, and z. We refer to S as the "action" and it is ~~a~~ not just a function, but a functional. The statement that, for example

$$\frac{d}{dt}S=0$$ means that the value of the action,

as encoding the laws of physics, is independent of when in time all particle trajectories are measured from.

See you Friday!