Lecture 21 Physics 101

Welcome to Friday! Please turn in homework!

Last lecture, we had introduced the concept of momentum, and generally expressed Newton's second law for a single object as

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}, \quad \text{where } \vec{p} = m\vec{v} \text{ is the momentum}$$

of the object of mass m with velocity $\vec{v}$. For a collection of objects, Newton's second law for the whole set can be found by simply summing together Newton's laws for each object individually. Doing so, we found:

$$\vec{F}_{ext} = \frac{d}{dt}\sum_i \vec{p}_i = \left(\sum_i m_i\right)\frac{d^2\vec{x}_{cm}}{dt}$$

Here, $\vec{F}_{ext}$ is the sum of external forces only, as all forces between objects of interest cancel by Newton's third law. $\vec{p}_i$ is the momentum of the i^{th} object, m_i is its mass, and $\vec{x}_{cm}$ is the position of the center-of-mass of the system:

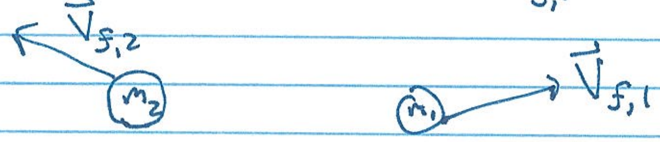
$$\vec{x}_{cm} = \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i}, \quad \text{where } \vec{x}_i \text{ is the position of the } i^{th} \text{ particle.}$$

Thus, momentum of a system is conserved (unchanged in time) if there are no external forces. Correspondingly, if there are no external forces, then the center-of-mass does not accelerate.

We've now introduced conservation laws for energy and momentum, which, by Noether's theorem, correspond to symmetries under time and spatial translations, respectively. These conservation laws are especially useful for analyzing collisions of two objects, such as two balls, cars, bouncing a ball on the ground, an asteroid hitting a planet, two galaxies colliding, etc. Sorry for getting a bit carried away, but collisions are <u>very</u> general physical phenomena, occurring essentially in any imaginable physical system. It's amazing that the <u>very</u> simple, yet enormously profound, ideas of conservation of energy and momentum are basically all we need to completely analyze <u>any</u> collision. Further <u>only</u> using the principles of conservation of energy and momentum to analyze any collision is an extremely strong test of these fundamental ideas.

So, with that prologue, it is useful for us to determine a taxonomy of different types of collisions, based on the relevant conservation laws. To set up, we will imagine colliding two objects of masses m_1 and m_2 with initial velocities $\vec{v}_{i,1}$ and $\vec{v}_{i,2}$:

then the masses collide, and after the collision, the masses have velocities $\vec{v}_{f,1}$ and $\vec{v}_{f,2}$:



First, as we have just discussed, if only forces internal to the mass 1 - mass 2 system act on the masses, then momentum is conserved:

$$\vec{F}_{ext} = 0 \Rightarrow m_1 \vec{v}_{i,1} + m_2 \vec{v}_{i,2} = m_1 \vec{v}_{f,1} + m_2 \vec{v}_{f,2}$$

Energy is always conserved, but the expression of energy might be useless or irrelevant for the collision of the masses (sound, heat, etc.). So, constraining to <u>useful</u> forms of energy, potential and kinetic, kinetic energy is conserved in the collision if <u>only</u> conservative forces act on the masses: ~~Further, those conservative~~

$$\vec{F}_{non\text{-}cons} = 0 \Rightarrow \frac{1}{2}m_1 v_{i,1}^2 + \frac{1}{2}m_2 v_{i,2}^2 = \frac{1}{2}m_1 v_{f,1}^2 + \frac{1}{2}m_2 v_{f,2}^2$$

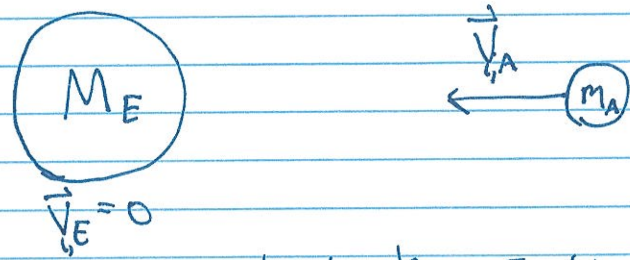
Now, we can consider the four possible force combinations and give names to different types of collisions:

	$\vec{F}_{non\text{-}cons} = 0$	$\vec{F}_{non\text{-}cons} \neq 0$
$\vec{F}_{ext} = 0$	Momentum, Kinetic Energy conserved "Elastic Collision"	Momentum Conserved "Inelastic Collision"
$\vec{F}_{ext} \neq 0$	Kinetic Energy Conserved	Neither Momentum nor Kinetic energy conserved

So, for analyzing a collision system, it just requires you to determine the relevant forces of the system to determine which conservation laws to use. We will mostly focus on Elastic and Inelastic collisions in this part of the course, as we are currently interested in momentum conservation. However, with Newton's second law or just conservation of energy we had analyzed collisions of the $\vec{F}_{ext} \neq 0$ type in previous weeks. For example, if friction is a relevant force, it is external to the colliding objects and non-conservative, but we know how to deal with it.

In the rest of this class, I want to analyze a particularly profound collision event in the history of Earth: the Cretaceous-Paleogene extinction event or Cretaceous-Tertiary extinction (K-T) that eliminated more than about 75% of extant life on Earth. The current widely accepted theory for this mass extinction event was the impact of an asteroid at what is now the Yucatán Peninsula about 66 million years ago, called the Chicxulub crater.

We will use the ideas of momentum conservation, energy conservation, and our theory of gravitation to determine the energy released by the asteroids' impact on Earth. Initially, before impact, the Earth and asteroid are in space as such:



We will assume that the Earth is at rest with respect to the space in which the collision occurs (the solar system). This of course isn't strictly true, ~~but~~ because the Earth is orbiting the sun, but we can imagine that the velocity of the asteroid is perpendicular to the velocity of Earth (i.e., radial toward the sun), so Earth's velocity is not relevant for the collision.

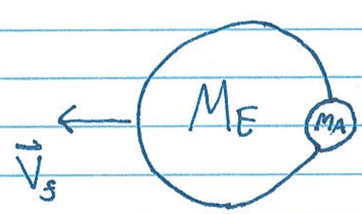
So with this set-up, the initial momentum of the Earth-asteroid system is

$$\vec{p}_i = \vec{p}_{i,E} + \vec{p}_{i,A} = m_A \vec{V}_{i,A}$$

and the initial total kinetic energy is

$$K_i = K_{i,E} + K_{i,A} = \frac{1}{2} m_A V_{i,A}^2$$

Now, what happens after the collision? The asteroid becomes embedded in the Earth, as such:



and now the Earth and asteroid travel with a common velocity $\vec{V}_f$.

Because the asteroid is stuck to the Earth, there must be some "sticky" force responsible for attaching the asteroid to the Earth. This sticky force is entirely internal to Earth and the asteroid, and there are no relevant external forces present, so, by our taxonomy, momentum of the Earth-asteroid system is conserved. That is, the final momentum is

$$\vec{p}_f = \vec{p}_{f,E} + \vec{p}_{f,A} = m_E \vec{v}_f + m_A \vec{v}_f = m_A \vec{v}_i,$$

where we have simply called $\vec{v}_{i,A} \equiv \vec{v}_i$.

In contrast with momentum conservation, the sticky force is <u>not</u> conservative: there is no well-defined potential energy ascribable to the sticky force. This is exactly analogous to the real, familiar force of sticky tape: it is ~~definitely~~ definitely a force as it can accelerate objects or support them against gravity, but has no potential energy, in the same way that friction does not. Therefore, kinetic energy is <u>not</u> conserved in this collision. However, it will be useful later to determine how much kinetic energy was lost, so we evaluate the final kinetic energy as:

$$K_f = K_{f,E} + K_{f,A} = \frac{1}{2}(m_E + m_A) v_f^2.$$

To continue, we will assume that the collision only happens in one-dimension, so we can drop the vectors in conservation of momentum. Then, the final velocity of the Earth-asteroid system is:

$$(m_E + m_A) v_f = m_A v_i \Rightarrow v_f = \frac{m_A}{m_E + m_A} v_i$$

Then, the final kinetic energy of the Earth-Asteroid system is

$$K_f = \frac{1}{2}(m_E + m_A) v_f^2 = \frac{1}{2} \frac{m_A^2}{m_E + m_A} v_i^2$$

From one perspective, we are done: given the masses of the Earth and the asteroid and the initial velocity of the asteroid, we can determine the final velocity of the system. However, I want to go further. Because the "sticky" force that embeds the asteroid in Earth is non-conservative, kinetic energy from the asteroid initially is lost in the collision, as heat, explosion of rock, deafening sound, etc. So, how much energy is released in this asteroid collision?

The energy released in the collision is simply the difference between the initial and final kinetic energies of the system:

$$E_{rel} = K_i - K_f = \frac{1}{2} m_A v_i^2 - \frac{1}{2}\frac{m_A^2}{m_E + m_A} v_i^2$$

$$= \frac{1}{2}\frac{m_E m_A}{m_E + m_A} v_i^2$$

The mass factor that appears in the final expression for the energy released is called the <u>reduced mass</u> and is denoted as:

$$\mu = \frac{m_E m_A}{m_E + m_A}$$

So, we can equivalently write the released energy as

$$E_{rel} = \frac{1}{2}\frac{m_E m_A}{m_E + m_A} v_i^2 = \frac{m_E}{m_E + m_A} \cdot \frac{1}{2} m_A v_i^2 = \frac{m_E}{m_E + m_A} K_i,$$

where K_i is the initial kinetic energy of the asteroid.

What could this kinetic energy be; or, rather, where did the asteroid get this kinetic energy from? From the gravitational force of the sun on the asteroid! To estimate the kinetic energy of the asteroid right before it hit Earth, let's imagine that it started at rest, very far away from the sun. Then, its initial energy would be 0: no kinetic energy and not gravitational potential energy. However, right before it hit Earth, it would have gained kinetic energy from losing gravitational potential energy by getting closer to the sun. Its total energy must still be 0, so

$$U_G + K_i = 0 = -\frac{G_N M_\odot m_A}{r_{E\odot}} + \frac{1}{2} m_A V_i^2$$

or that the initial kinetic energy is

$$K_i = \frac{1}{2} m_A V_i^2 = \frac{G_N M_\odot m_A}{r_{E\odot}},$$

where $M_\odot$ is the mass of the sun and $r_{E\odot}$ is the radius of Earth's orbit around the sun. So, the energy released by the asteroid hitting Earth is

$$E_{rel} = \frac{m_E}{m_E + m_A} \frac{G_N M_\odot m_A}{r_{E\odot}}$$

Now, we need to estimate the mass of the asteroid, m_A. It's estimated that the diameter of the asteroid, inferred from the Chicxulub crater's size, was about 10 km. This is about a factor of 1000 times smaller than the size of Earth. The amount of mass (= stuff) in an object is determined by its volume, which is proportional to the cube of the diameter. So, if the asteroid has a diameter that is a thousand times smaller than Earth, then its volume is smaller by the cube of this, or a factor of 1 billion! So, we estimate the mass of the asteroid to be a billion times smaller than the mass of Earth:

$$m_A = 10^{-9} m_E$$

With this identification, the ratio factor in front of the released energy is very nearly just 1:

$$\frac{m_E}{m_E+m_A} = \frac{1}{1+{m_A}/{m_E}} = \left(1+10^{-9}\right)^{-1} = 1-10^{-9}+\ldots,$$

using the binomial expansion. So, for our estimate, we will just set it to 1.

The energy released by the asteroid is then

$$E_{rel} \cong \frac{G_N M_\odot m_A}{r_{E\odot}}, \text{ where}$$

$$G_N = 6.67\times10^{-11}\ kg^{-1}\,m^3\,s^{-2},\quad M_\odot = 2\times10^{30}\ kg,$$

$$m_A \cong 10^{-9} M_E \cong 6\times10^{15}\ kg,\quad r_{E\odot} = 1.5\times10^{11}\ m.$$

Plugging these numbers in, we find the energy released in the asteroid collision to be

$$E_{rel} \cong 5\times10^{24}\ J.$$

This is an exceedingly large amount of energy, enough to wipe out most life on Earth. It is estimated that Mt. St. Helens released about 10^{18} J of energy in its explosion in 1980. The asteroid that created the Chicxulub crater would have been like a million Mt. St. Helens events simultaneously.

That's it for today! Have a good weekend!