Physics 101 Lecture 22

Hope you had a good weekend! Please turn in homework!

We are going to continue our discussion of momentum conservation today, focusing on momentum conservation in multiple dimensions. Again, for a system of n particles or objects, we can write Newton's second law in terms of momentum as

$$\vec{F}_{ext} = \frac{d}{dt}\sum_{i=1}^{n}\vec{p}_i = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n\right)$$

where $\vec{p}_i$ is the momentum of particle i, and $\vec{F}_{ext}$ is the sum of forces external to the n particles. If there is no (relevant) net external force then the time derivative of the sum of momentum is 0:

$$\vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}\sum_{i=1}^{n}\vec{p}_i = 0$$

or that the sum of momentum is conserved, unchanged, in time:

$$\sum_{i=1}^{n}\vec{p}_i = \text{constant.}$$

This is a vector equation so the sum of each component of momentum is conserved individually if $\vec{F}_{ext} = 0$:

$$\sum_{i=1}^{n} p_{i,x} = \text{constant}, \quad \sum_{i=1}^{n} p_{i,y} = \text{constant, etc.}$$

where $p_{i,x}$ and $p_{i,y}$ are the x- and y-components of the momentum of particle i, respectively. We had previously exploited momentum conservation to analyze collisions that exclusively occurred in one-dimension; in this lecture, we will study the constraints that momentum conservation in two-dimensions imposes on systems with multiple particles. We will focus on two-dimensions rather than three-dimensions because going to three-dimensions mostly just adds complication and additional bookkeeping.

To set the stage for this lecture, I'd like to study the physics of the Reed Reactor. How many of you work at the reactor, or are in training? How many of you ever applied to Reed in part because of the Reactor? How many are familiar with the reactor at all? Like any nuclear reactor, the Reed reactor is powered at its core by a radioactive element that decays to other particles after a characteristic time called the half-life. The Reed reactor uses plutonium as its core, but the amount of plutonium is tiny and is much too small to output a useful amount of energy, or to ever be a threat to campus. Nevertheless, there is very interesting physics in nuclear decay, and in this lecture we will explore one aspect through momentum conservation.

A complete, or even simply honest, discussion of nuclear decay requires concepts of both special relativity and quantum mechanics, but we'll be able to simplify our discussion enough to not need their details. Fundamentally, nuclear decay is a consequence of the decay of the neutron, one of the constituents of atomic nuclei, along with protons. A neutron is observed to decay to a proton and an electron, and we denote this decay as:

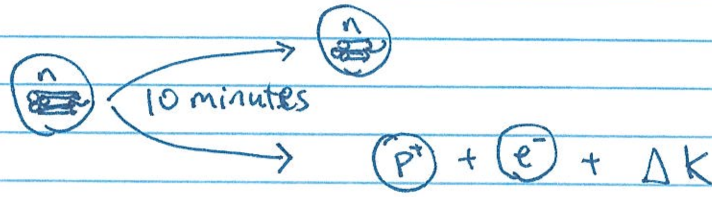
$n \to p^+ + e^-$, where we denote the neutron as n, the proton is p^+, and the electron is e^-.

The superscripts denote the electric charge of the proton and electron; the neutron is electrically neutral. You'll be introduced to these concepts next semester. In addition to the neutron producing a proton and an electron, kinetic energy is also produced, that pushes the proton and electron apart. Let's call this kinetic energy ΔK. So, our model for neutron decay is the following. We initially have a neutron hanging out in space at rest, $\vec{v}=0$:

n

As a model for decay and kinetic energy release, we imagine that there is a little time bomb in the neutron:

After the half-life time of the neutron elapses, about ten minutes, there is a 50% change that the neutron decays and a 50% chance it does not:



If the neutron decayed, then this model/hypothesis for neutron decay makes concrete predictions for the kinetic energy of the electron that we can test in experiment.

If the neutron is at rest when it decays, then it has no momentum. The decay/explosion of the neutron into a proton and electron exclusively involves forces <u>internal</u> to the proton-electron system, so momentum <u>is conserved</u>. This means that the sum of proton and electron momentum is 0:

$$m_p \vec{v}_p + m_e \vec{v}_e = 0.$$

As mentioned earlier, we will study this decay in two dimensions, so we can write this out in components as:

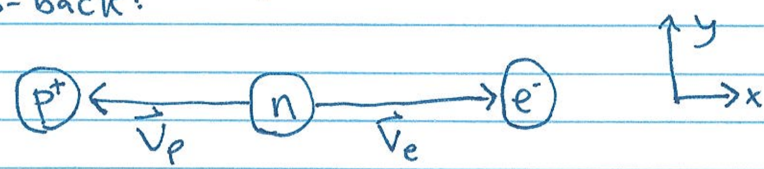
$$m_p v_{p,x} + m_e v_{e,x} = 0, \quad m_p v_{p,y} + m_e v_{e,y} = 0.$$

Further, the neutron decay produces a kinetic energy of ΔK, carried by the proton and electron. This means that

$$\Delta K = \frac{1}{2} m_p v_p^2 + \frac{1}{2} m_e v_e^2 = \frac{1}{2} m_p \left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2} m_e \left(v_{e,x}^2 + v_{e,y}^2\right),$$

where we have written the kinetic energy out in components on the right. Now, we would like to massage these equations to determine the energy of the electron individually.

The first thing to note is that, although we have expressed momentum and energy with <u>both</u> components of the proton's and electron's velocity vectors, we can set one of the components to 0. Momentum conservation requires that the velocity of the proton and electron are back-to-back:



So, we can just orient our axes such that there is no y-component of velocity: $v_{p,y} = v_{e,y} = 0$. With this orientation, the x-component of velocity is simply the total speed of the proton and electron:

$$v_{p,x} = v_p, \quad v_{e,x} = v_e.$$

Then, conservation of momentum and energy are:

$$m_p v_p + m_e v_e = 0, \qquad \Delta K = \frac{1}{2} m_p v_p^2 + \frac{1}{2} m_e v_e^2$$

Momentum conservation implies that: $v_p = -\frac{m_e}{m_p} v_e$,

and plugging this into energy conservation, we find:

$$\Delta K = \frac{1}{2} m_p \left(\frac{m_e}{m_p} v_e\right)^2 + \frac{1}{2} m_e v_e^2 = \frac{1}{2} \frac{m_e^2 + m_p m_e}{m_p} v_e^2$$

$$= \frac{1}{2} m_e v_e^2 \cdot \left(\frac{m_e + m_p}{m_p}\right) \equiv \frac{m_e + m_p}{m_p} K_e$$

Solving for the kinetic energy of the electron, we find

$$\frac{1}{2} m_e V_e^2 = K_e = \frac{m_p}{m_p + m_e} \Delta K$$

The mass of the proton is about 2000 times that of the electron, so we can approximate the mass ratio factor:

$$\frac{m_p}{m_p + m_e} = \frac{1}{1 + m_e/m_p} \cong \left(1 + \frac{m_e}{m_p}\right)^{-1} \simeq 1 - \frac{m_e}{m_p} \cong 1 - \frac{1}{2000} \cong 1.$$

So, the kinetic energy of the electron would just be approximately the kinetic energy released by the neutron decay:

$$K_e \cong \Delta K.$$

ΔK is a fixed value of about 1.2×10^{-13} J or 780 keV (that is, kilo-electron volts), so this model predicts that the kinetic energy of every electron produced from neutron decay always carries this kinetic energy. So, we test this prediction by preparing a large number of neutrons, let them decay, and measure the kinetic energy of the produced electron from decay. If we always see the electron carry $K_e = \Delta K \cong 1.2 \times 10^{-13}$ J, then we gain evidence for the model $n \to p^+ + e^-$ decay.

If one does this experiment, this is not what is found! Instead of the electron always carrying kinetic energy ΔK, it is found that the electron carries kinetic energy

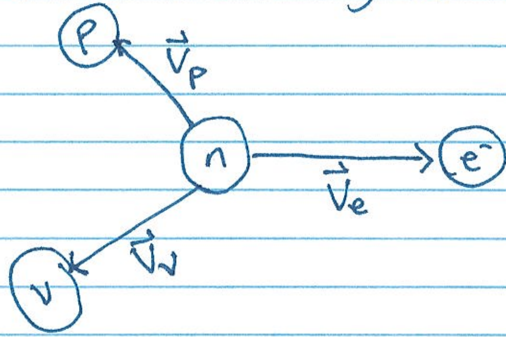
$$K_e \in [0, \Delta K], \text{ bounded from above by } \Delta K.$$

So, our hypothesis was incorrect. What is the simplest thing we can do to modify it? We could throw out momentum and energy conservation, but that is very dramatic because we have so much evidence for their conservation otherwise. Also, by Occam's razor, the simplest explanation is typically the correct one, so we don't want to consider non-conservation unless we are absolutely forced to. Well, in the decay of the neutron, we observe the proton and the electron decay products. However, what if there was another, third, decay product that we could not observe directly? How would that affect the kinetic energy and momentum that the electron carried?

Let's now hypothesize that the neutron decays to three particles, referred to as a <u>three-body decay</u>:

$n \to p^+ + e^- + \nu$, where the third particle is denoted with the Greek letter nu, ν, and is called a <u>neutrino</u> (story to follow). What do conservation of momentum and energy look like now?

We illustrate the decay of the neutron now as:



with conservation of momentum and energy:

$$m_e\vec{v}_e + m_p\vec{v}_p + m_\nu\vec{v}_\nu = 0, \quad \Delta K = \tfrac{1}{2}m_e v_e^2 + \tfrac{1}{2}m_p v_p^2 + \tfrac{1}{2}m_\nu v_\nu^2.$$

(I should say that these are not the correct conservation laws for this decay because the neutrino is travelling at essentially the speed of light. However, it is sufficient to illustrate the interesting physics.)

Unlike the two-body decay we had studied earlier, this three-body decay is <u>not</u> confined to a line, so we have to consider a more general, two-dimensional, conservation law. We can, however, still rotate our axes to be convenient; we will choose to align the electron velocity/momentum with the x-axis. Then, the conservation laws in components are:

$$m_e v_e + m_p v_{p,x} + m_\nu v_{\nu,x} = 0\,, \quad m_p v_{p,y} + m_\nu v_{\nu,y} = 0$$

$$\Delta K = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p \left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2} m_\nu \left(v_{\nu,x}^2 + v_{\nu,y}^2\right)$$

Now, we have 3 conservation laws, but 5 unknowns ($v_e, v_{p,x}, v_{p,y}, v_{\nu,x}, v_{\nu,y}$), so we cannot uniquely solve the system of equations. However, we can simplify it and eliminate dependence on the neutrino velocity, $\vec{v}_\nu$. From the conservation laws for momentum, we have

$$v_{\nu,x} = -\frac{m_e v_e + m_p v_{p,x}}{m_\nu}\,, \quad v_{\nu,y} = -\frac{m_p}{m_\nu} v_{p,y}$$

Plugging these expressions into the conservation of energy, we find:

$$\Delta K = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p \left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2} m_\nu \left(\frac{(m_e v_e + m_p v_{p,x})^2}{m_\nu^2} + \frac{m_p^2}{m_\nu^2} v_{p,y}^2\right)$$

or that

$$\Delta K = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p \left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{1}{2} \frac{m_e^2}{m_\nu} v_e^2 + \frac{1}{2} \frac{m_p^2}{m_\nu} \left(v_{p,x}^2 + v_{p,y}^2\right) + \frac{m_e m_p}{m_\nu} v_e v_{p,x}$$

$$= \frac{1}{2} m_e v_e^2 \left(1 + \frac{m_e}{m_\nu} + \frac{m_p}{m_\nu} \frac{v_{p,x}}{v_e}\right) + \frac{1}{2} m_p v_p^2 \left(1 + \frac{m_p}{m_\nu}\right),$$

which has no residual dependence on the unobservable neutrino velocity! We can solve for $K_e = \frac{1}{2} m_e v_e^2$ and find

$$K_e = \frac{1}{2} m_e v_e^2 = \frac{\Delta K - K_p\left(1 + \frac{m_p}{m_\nu}\right)}{1 + \frac{m_e}{m_\nu} + \frac{m_p}{m_\nu} \frac{v_{p,x}}{v_e}}\,, \quad \text{where } K_p = \frac{1}{2} m_p v_p^2.$$

Now, unlike the case when we hypothesized that the neutron decayed exclusively to the proton and electron, the kinetic energy of the electron has a range of possible values, just as we measure in experiment. For example, the speed of the electron can vanish: $v_e = 0$, and momentum and energy still be conserved through the proton and neutrino. Thus, conservation laws can be exploited to determine and identify new particles you can't in otherways directly observe!

This conservation of momentum and energy (and angular momentum) argument for neutron decay, or radioactive decay more generally, was used in the early 1930s to postulate the neutrino. In the 19-teens, 20s, and 30s, people like the Curies and Lise Meitner identified radioactivity in heavy, unstable, isotopes, and even constructed a theory for their mechanism. However, it was observed that the conservation laws had problems connecting theory to experinental data. To rectify it, Wolfgang Pauli postulated the existence of a new particle that was also produced in radioactive decay which he called the "neutron", but we now call the "neutrino" ("little neutral one" in Italian; coined by Enrico Fermi). Pauli had been invited to a conference on radioactivity in Tübingen, Germany, in the early 1930s when he had the idea. Unfortunately, his attendance was required at a ball in Zürich, Switzerland, at the same time, so couldn't attend the conference. However, in lieu of attendance Pauli wrote a letter to Lise Meitner who was at the conference in which he laid out his idea for the "neutrino". Pauli addressed the attendees of the conference in the letter as:

Dear radioactive ladies and gentlemen,

and further apologized for his absence!