Lecture 23 Physics 101

Welcome back to physics on this lovely Wednesday! Please turn in homework.

I hope that I have impressed upon you the importance of momentum conservation and how it has very ~~imp~~ relevant consequences for our world. In this lecture, we're going to start pivoting from momentum conservation to the discussion of the dynamics of rotations. Our first step will be to, as always, revisit Newton #2 written in a form we had introduced last week. For a system of n masses, we had shown that Newton's second law could be written as:

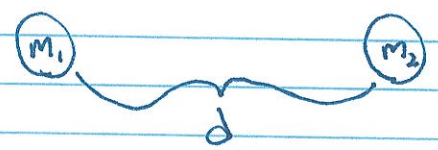
$$\vec{F}_{ext} = \left(\sum_{i=1}^{n} m_i\right) \frac{d^2 \vec{x}_{cm}}{dt^2},$$

where $\sum_{i=1}^{n} m_i = m_1 + m_2 + \cdots + m_n$ is the sum of masses in the system and $\vec{x}_{cm}$ is the position of the center-of-mass:

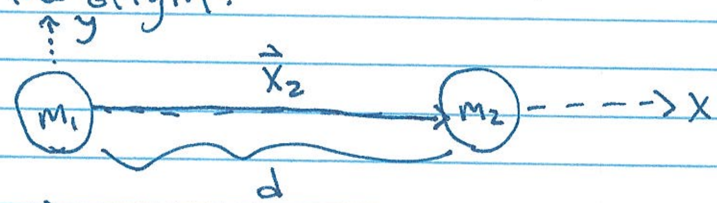
$$\vec{x}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{x}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 \vec{x}_1 + m_2 x_2 + \cdots + m_n \vec{x}_n}{m_1 + m_2 + m_3 + \cdots + m_n}.$$

We had been introduced to this center-of-mass in one of the conferences, as well, but we said nothing about it, other than two mutually-gravitationally bound masses orbit their common center-of-mass. Why is that the case and what consequences does the center-of-mass have for systems that in general are composed of many individual masses? Let's find out!

Let's first just study a system that consists of two masses, m_1 and m_2, separated by a distance d:



To calculate their center-of-mass, we need to know the position vectors of the masses, which in turn requires setting up a coordinate system. As I have always said, life cannot imitate art in physics, which means that this coordinate system is ~~sy~~ simply a tool to be able to talk concretely about the masses, but their physical location, and hence the physical location of their center-of-mass is the same place, regardless of the coordinates used. So, we simply pick convenient coordinates with, say, mass 1 located at the origin:

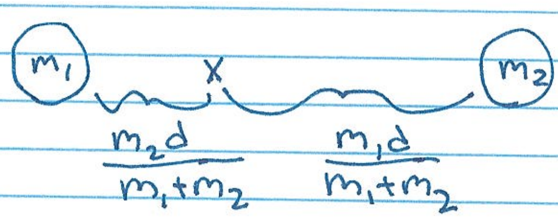


$$\vec{x}_1 = 0, \quad \vec{x}_2 = d\hat{i}$$

Then, the center-of-mass location in these coordinates is:

$$\vec{x}_{cm} = \frac{\vec{x}_1 m_1 + \vec{x}_2 m_2}{m_1 + m_2} = \frac{0 \cdot m_1 + d\hat{i} \cdot m_2}{m_1 + m_2} = \frac{m_2 d}{m_1 + m_2}\hat{i}$$

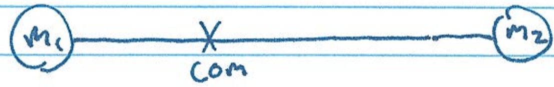
Note that m_1 and m_2 are positive, so $\frac{m_2}{m_1+m_2} < 1$, which means that the location of the center-of-mass lies <u>between</u> the masses:



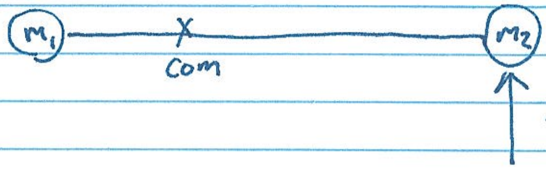
Of course, the sum of the distances of ~~the~~ each mass to their combined center-of-mass is d:

$$d = \frac{m_2 d}{m_1 + m_2} + \frac{m_1 d}{m_1 + m_2}$$

Now, let's imagine that the two masses are connected by a rigid, massless rod, like so:



Now, with this set up, let's imagine exerting a force upward on mass 2, like so:



what does this mass-rod system do, immediately after the force is applied?

To answer this question, let's consider the Newton's law we derived for a system of masses. First the free-body diagram for this system is:

$$\Rightarrow \quad \vec{F} = (m_1 + m_2)\,\vec{a}_{cm}$$

where we imagine the force acting exclusively at the center-of-mass. So, the acceleration of the center-of-mass is:

$$a_{cm} = \frac{F}{m_1 + m_2} \text{ upward.}$$

Address later!

~~That's the center-of-mass: what is the direction of acceleration of mass m_1 in this set-up?~~

~~a) $\uparrow \vec{a}_1$ b) $\vec{a}_1 \nearrow$ c) $\longrightarrow \vec{a}_1$ d) $\vec{a}_1 = 0$~~

~~Talk with your neighbors for a minute!~~

~~To answer this question, let's think about what the rod is doing. As the rod is rigid, it definitely exerts a force: forces on mass 1 has to transmit to mass 2 instantaneously so that the rod stays rigid and the distance between the masses stays fixed. The rod is massless similar to how ropes were when we considered pulleys, so whatever force it exerts on mass 1 has to be opposite to that on mass 2, to~~

ensure that the rod neither stretches nor compresses. Further, it doesn't matter where on this system we consider the force exerted; on mass 1, somewhere on the rod, etc., the center-of-mass always accelerates with this rate. What response the individual masses have depends on where the force is applied, but the center-of-mass just accelerates according to the net force.

This can be exploited to great effect. First let's consider putting this mass-rod system near the surface of the Earth. Now, there is a net external force, namely, gravity. By our work thus far, this force acts at the center-of-mass:

$$\vec{F}_g = -(m_1+m_2)g\hat{j} = (m_1+m_2)\vec{a}_{cm}, \text{ or that } \vec{a}_{cm} = -g\hat{j}.$$

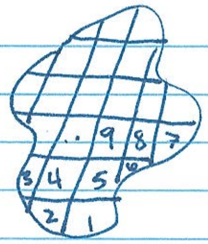
So, if we just let this thing fall, the center-of-mass would accelerate at g. Conversely, to hold the object up, we need to apply some normal force that prevents the center-of-mass from accelerating downward. As long as we do this, the object will not fall! So we only need to hold it up by the center-of-mass!

Let's see how this works in a demo!

Now, this two-mass system is just the starting point; we would like to determine the center-of-mass for an arbitrary system of masses. We will consider some blob as the system of interest:

Let's assume that the blob is just two dimensional for simplicity. How do we calculate its center-of-mass?

As often in this game, let's break it up into many small masses and sum them up. So, we will consider



where the small submasses are labeled. The total mass M of the object is:

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

Further, let's introduce A_i as the area of the i^{th} piece. Then, we call the density σ_i the ratio of mass to area:

$$\sigma_i = \frac{m_i}{A_i}$$

Now, this is two dimensional, so we can specify the point i by its x and y coordinates:

$$\sigma_i \equiv \sigma(x,y) = \frac{m(x,y)}{A(x,y)}$$

Now, if the areas are taken to be very small rectangles with sides of length dx and dy, the area is

$A(x,y) = dx\,dy$, and for a small area, mass is small

so we denote $m(x,y) \equiv dm$. Schematically, we then have that the density is:

$$\sigma(x,y) = \frac{dm}{dx\,dy}$$

Now, the total mass is simply the sum of all these small masses. In the limit that the masses get infinitesimally small, the sum transforms into an integral

$$M = \sum_i m_i \rightarrow \int dm = \iint \frac{dm}{dx\,dy} \cdot dx\,dy = \iint \sigma(x,y)\,dx,dy$$

On the right, we have introduced a double integral over the density to calculate mass. A double integral is just two nested integrals: do x integral first, assuming y is constant, then integrate over y. So, our expression for the mass of a complicated shape is

$$M = \iint \sigma(x,y)\, dx\, dy$$

Now, to calculate the center-of-mass of this object, we weight each tiny mass by its position vector and divide by the total mass:

$$\vec{X}_{cm} = \frac{1}{M}\int dm\, \vec{r} = \frac{1}{M}\iint \sigma(x,y)\, \vec{r}\, dx\, dy$$

$$= \frac{1}{\iint \sigma(x,y)\, dx\, dy} \iint \sigma(x,y)\, (x\hat{i} + y\hat{j})\, dx\, dy$$

To integrate over the vector of position $\vec{r}$, the unit vectors $\hat{i}$ and $\hat{j}$ are constants, unaffected by integration. They need to be kept around to determine the vector position of the center-of-mass.

Now, with these observations, we are going to do a few demos. First, consider an arbitrary shaped mass. I hang the mass from a nail and it ~~swings~~ swings freely until it comes to rest. With respect to the nail, where is the center-of-mass?

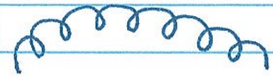
a) right of the nail b) left of the nail

c) below the nail d) above the nail

Talk to your neighbors for a bit!

Let's test it out!

Now, let's consider another system. Let's take this croquet mallet, and on it, someone has nicely put LED lights at the location of the center-of-mass of the mallet. I am going to turn off the lights in a second and throw the mallet across the room. What will the trajectory of the lights be?

a) 

b) 

c) lights do not work when moving

d) 

By Grabthar's Hammer, let's try this out!

To close this mostly demonstrative lecture, we have seen the beauty and power of working with the center-of-mass. However, there's clearly interesting dynamics for points on the object that are <u>not</u> the center-of-mass. What is this dynamics? How can we understand it? More next time...