Lecture 24 Physics 101

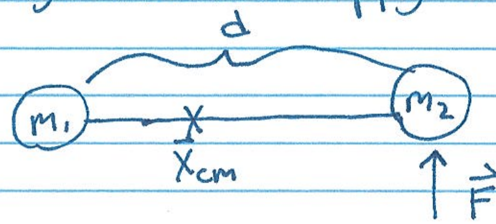
Welcome to Friday! Please turn in homework!

Last lecture, we demonstrated and discussed properties of the center-of-mass of an object that has extended or irregular structure. Regardless of what force exert on an object or how, the center-of-mass of that object accelerates simply according to Newton's second law:

$$\vec{F}_{ext} = M \frac{d^2\vec{x}_{cm}}{dt^2},$$

where M is the total mass of the object. Also as we observed on Wednesday, this clearly isn't the whole story. When I threw the mallet to Owen, you indeed saw that the center-of-mass traveled in a parabolic trajectory, as expected from projectile motion. However, as the center-of-mass was traveling as a projectile, the head and handle of the mallet rotated end-over-end about the center-of-mass, clearly a non-projectile motion. How do we model this motion and describe the more general motion of the entire mallet, not just its center-of-mass?

Let's start this discussion with a physical set-up we mentioned briefly in the previous lecture. Let's consider two masses separated by distance d by a rigid rod and apply vertical force $\vec{F}$ to mass m_2:



From our earlier analysis, we know that applying this force accelerates the center-of-mass:

$$\vec{F} = (m_1+m_2)\vec{a}_{cm} \quad \text{or that} \quad \vec{a}_{cm} = \frac{F}{m_1+m_2}\hat{j}$$

So, the center-of-mass just accelerates vertically according to the magnitude of $\vec{F}$ and the sum of the masses. What about the accelerations of masses 1 and 2 individually?

Let's break apart Newton's law as an explicit sum over masses 1 and 2's accelerations:

$$\vec{F} = (m_1 + m_2)\,\vec{a}_{cm} = (m_1 + m_2)\,\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

$$= m_1\vec{a}_1 + m_2\vec{a}_2$$

The sum of accelerations $\vec{a}_1$ and $\vec{a}_2$ is constrained by the acceleration of the center-of-mass, so let's express the individual accelerations as:

$$\vec{a}_1 = \vec{a}_{cm} + \Delta\vec{a}_1 \quad , \quad \vec{a}_2 = \vec{a}_{cm} + \Delta\vec{a}_2 \, ,$$

for some accelerations $\Delta\vec{a}_1$, $\Delta\vec{a}_2$. So far, this is just a tautology; we have done nothing but shift our notation. However, now we can find a simple relationship between $\Delta\vec{a}_1$ and $\Delta\vec{a}_2$ from Newton's second law:

$$\vec{F} = (m_1 + m_2)\,\vec{a}_{cm} = m_1(\vec{a}_{cm} + \Delta\vec{a}_1) + m_2(\vec{a}_{cm} + \Delta\vec{a}_2)$$

or that $m_1\Delta\vec{a}_1 + m_2\Delta\vec{a}_2 = 0$. We'll use this relationship to relate $\Delta\vec{a}_1$ to $\Delta\vec{a}_2$:

$$\Delta\vec{a}_1 = -\frac{m_2}{m_1}\,\Delta\vec{a}_2 \, .$$

So, the accelerations are:

$$\vec{a}_1 = \vec{a}_{cm} - \frac{m_2}{m_1}\,\Delta\vec{a}_2 \quad , \quad \vec{a}_2 = \vec{a}_{cm} + \Delta\vec{a}_2 \, .$$

There are more constraints we can exploit.

The fact that the masses are connected by a perfectly rigid rod constrains their motion about the center-of-mass and relative to one another. The center-of-mass, from this applied force moves exclusively vertically. Because of the rigidity of the rod, mass 2, for example, remains a ~~fix~~ fixed distance away from the center-of-mass. However, it can, in principle, have any orientation about the center-of-mass. So, what type of motion is constrained to be a fixed distance from a point, but have any orientation? Circular motion!

Therefore, masses 1 and 2 travel in a circular orbit about the center-of-mass as the center-of-mass accelerates vertically! So somehow this circular motion is accounted for in the $\Delta\vec{a}_2$ acceleration we have yet to find. Again let's use rigidity to find more constraints on this circular motion. Because the rod is pefectly rigid, the masses m_1 and m_2 are always on opposite sides of their respective circlular orbits: that is, they necessarily orbit with the same angular velocity. If they had different angular velocities, then they would no longer be antipodal, and the rod must crumple, but that can't happen.

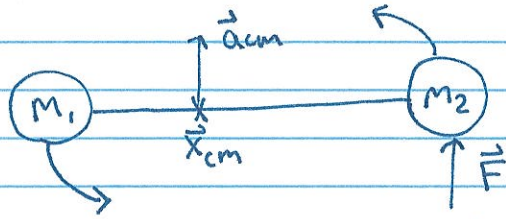
Angular velocity ω is defined as the first time derivative of the angle θ through which an object travels:

$$\frac{d\theta}{dt} = \omega.$$

We can take a second derivative to identify the angular acceleration:

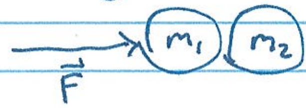
$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \alpha.$$

Now, if the angular velocities of masses 1 and 2 are identical, $\omega_1 = \omega_2$, then so too must their angular accelerations be: $\alpha_1 = \alpha_2$. The force $\vec{F}$ works to angularly accelerate the masses in the same direction of rotation:



common angular accleration

So, we can imagine that $\vec{F}$ is responsible for the rotation of m_1 and m_2 in the counterclockwise direction, in the same way that it would accelerate them linearly if it pushed m_1 and m_2 when touching; i.e.,



→ common acceleration

To rotate m_1 and m_2 about the center-of-mass, then $\vec{F}$ has to push against the inertia of both masses, to get them both to rotate. Just like in the linear case where the two masses would have a common linear acceleration and the force would push against the combined mass of the two blocks;

$$\vec{F} = (m_1 + m_2)\vec{a} \quad \text{(linear)}$$

the fact that in this rotating case the two masses have the same angular acceleration suggests a nice way to interpret.

The force $\vec{F}$ provides a tangential acceleration of the two masses in their orbit about the center-of-mass. That is, Newton's second law implies:

$$\vec{F} = \cancel{m_1 \vec{a}_{tan,1} +} m_2 \vec{a}_{tan,2} \quad \text{or} \quad F = \cancel{m_1 a_{tan,1} +} m_2 a_{tan,2},$$

where we can drop the vectors because rotation is occurring in one plane (the plane of the page).

Note that only the acceleration of mass 2 appears because $\vec{F}$ only directly acts on mass 2. Tangential acceleration is related to angular acceleration by a factor of radius from the center of the orbit. What is this relevant radius, to relate

$$a_{tan,2} = R\alpha \,?$$

Right at the instant when F is applied, only mass 2 moves; mass 1 remains stationary (for an instant). So, at that moment mass 2 is orbiting mass 1, a distance d away. Therefore, we initially have

$a_{tan,2} = d\alpha$, so that Newton's law is simply $F = m_2 d\alpha$. Now, one can use this to go back and solve for the unknown $\Delta\vec{a}_2$, but we won't do that here (though I encourage you to do so at home!). I want to massage this expression into another form that exclusively uses information about the rotation of the masses about their common center-of-mass.

Note the string of identities:

$$F\frac{m_1 d}{m_1+m_2} = \cancel{\frac{m_1 d m_2}{m_1+m_2} d\alpha} = \frac{m_1 m_2 d}{m_1+m_2} d\alpha$$

$$= \left[m_1 \frac{m_2^2 d^2}{(m_1+m_2)^2} + m_2 \frac{m_1^2 d^2}{(m_1+m_2)^2}\right]\alpha$$

$$= \left(m_1 R_1^2 + m_2 R_2^2\right)\alpha = F R_2$$

Here, R_1 and R_2 are the distances from the center-of-mass to the respective masses. The factor of R_2 that multiplies F is the distance from the applied force $\vec{F}$ to the center-of-mass.

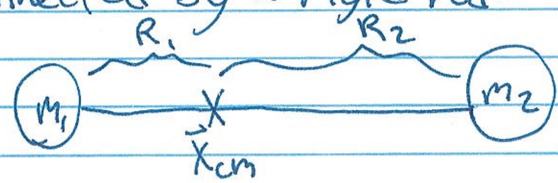
I emphasize that this is nothing more than Newton's second law, but expressed in a way useful for rotations. The mass-times-radius factors are called <u>moments of inertia</u> I defined as

$I = MR^2$, where M is the mass of the object and R is the distance to the rotation axis. Apparently, we have

$$FR_2 = (I_1 + I_2)\alpha, \text{ which looks alot like}$$

$F = (m_1 + m_2)a$ for common linear motion!

Before ending for the day, I want to study one more system. Let's again consider the two masses connected by a rigid rod:



In this system, we will have the center-of-mass travel with constant velocity $\vec{V}_{cm} = V_{cm}\hat{j}$, and the masses orbit the center-of-mass at angular frequency ω. Let's calculate the kinetic energy of this system. The kinetic energy is simply:

$K = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$, but we need to find $\vec{V}_1$ and $\vec{V}_2$ given the data of the problem.

Let's first consider mass 1. We can break its velocity into the linear component and rotational component. The linear component is simply the center-of-mass velocity:

$$\vec{V}_{1,lin} = V_{cm}\hat{j}.$$

For the rotational component, we had studied this weeks ago and can express the ~~ea~~ tangential velocity as:

$$\vec{V}_{1,rot} = -\omega R_1 \sin\omega t\,\hat{i} + \omega R_1 \cos\omega t,$$

for example. Then, the total velocity of mass 1 is:

$$\vec{V}_1 = \left(-\omega R_1 \sin\omega t\right)\hat{i} + \left(V_{cm} + \omega R_1 \cos\omega t\right)\hat{j}$$

and its square is:

$$|\vec{V}_1|^2 = \omega^2 R_1^2 \sin^2\omega t + V_{cm}^2 + \omega^2 R_1^2 \cos\omega t + 2V_{cm}\omega R_1 \cos\omega t$$

$$= V_{cm}^2 + \omega^2 R_1^2 + 2V_{cm}\omega R_1 \cos\omega t.$$

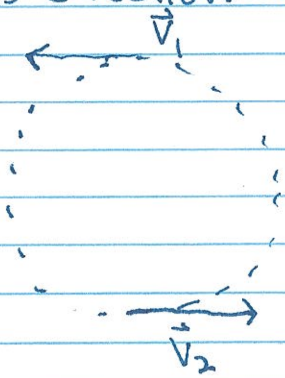
Then, the kinetic energy of mass 1 is:

$$K_1 = \frac{1}{2} m_1 V_{cm}^2 + \frac{1}{2} m_1 R_1^2 \omega^2 + m_1 V_{cm}\omega R_1 \cos\omega t$$

Let's now do the same thing for mass 2. ~~Because~~ Mass 2's linear velocity is still the center-of-mass velocity:

$$\vec{V}_{2,lin} = V_{cm}\,\hat{j}.$$

For the rotational component of mass 2's velocity, note that its direction must be opposite to that of mass 1:



and the magnitude of v_2 is determined by the angular velocity ω and its distance from the center-of-mass:

$$\vec{V}_{2,rot} = \omega R_2 \sin\omega t\,\hat{i} - \omega R_2 \cos\omega t\,\hat{j}$$

Then, the total velocity of mass 2 is:

$$\vec{v}_2 = (\omega R_2 \sin\omega t)\hat{i} + (V_{cm} - \omega R_2 \cos\omega t)\hat{j}$$

and its square is:

$$|\vec{v}_2|^2 = \omega^2 R_2^2 \sin^2\omega t + V_{cm}^2 + \omega^2 R_2^2 \cos^2\omega t - 2V_{cm}\omega R_2 \cos\omega t$$

$$= V_{cm}^2 + \omega^2 R_2^2 - 2V_{cm}\omega R_2 \cos\omega t$$

Then, the kinetic energy of mass 2 is:

$$K_2 = \frac{1}{2} m_2 V_{cm}^2 + \frac{1}{2} m_2 R_2^2 \omega^2 - m_2 V_{cm}\omega R_2 \cos\omega t.$$

The total kinetic energy is the sum:

$$K = K_1 + K_2 = \frac{1}{2}(m_1+m_2)V_{cm}^2 + \frac{1}{2} m_1 R_1^2 \omega^2 + \frac{1}{2} m_2 R_2^2 \omega^2$$
$$+ V_{cm}\omega \cos\omega t (m_1 R_1 - m_2 R_2).$$

Note that $m_1 R_1 = m_2 R_2 \Rightarrow m_1 \frac{m_2 d}{m_1+m_2} = m_2 \frac{m_1 d}{m_1+m_2}$

so the weird final term vanishes!

We can therefore write the kinetic energy of this system as:

$$K = \frac{1}{2}(m_1+m_2)V_{cm}^2 + \frac{1}{2}(I_1 + I_2)\omega^2$$

where $I_1 = m_1 R_1^2$, $I_2 = m_2 R_2^2$, the moments of inertia of the two masses. Cool!

Have a good weekend!