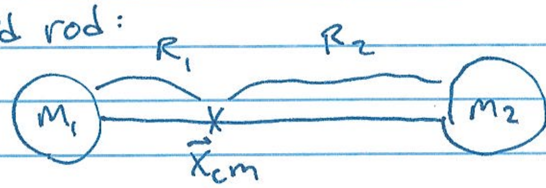
Lecture 25 Physics 101

Welcome back on this fine morning! Please turn in homework!

On Friday, we studied the dumbbell system of two masses m_1 and m_2 connected together by a massless, rigid rod:

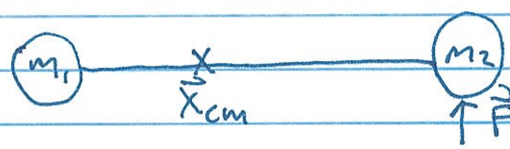


The center-of-mass of this system is identified and the distances of each mass to the center-of-mass is:

$$R_1 = \frac{m_2 d}{m_1 + m_2}, \quad R_2 = \frac{m_1 d}{m_1 + m_2},$$

where the separation distance of the masses is d. We had considered exerting a force upward on mass 2, which had one consequence of accelerating the center-of-mass:

$$\vec{F} = (m_1 + m_2)\vec{x}_{cm},$$



but also had the consequence of rotating the two masses about the center-of-mass. By considering the net force on mass 2 exclusively, we demonstrated that its Newton's second law could be written as:

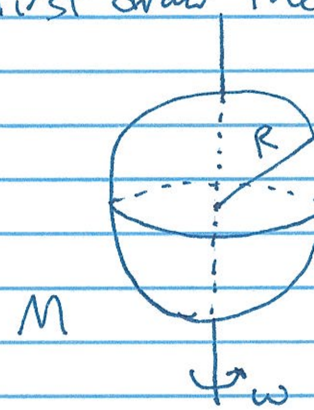
$$F R_2 = \left(m_1 R_1^2 + m_2 R_2^2 \right) \alpha,$$

where α is the angular acceleration of the two masses about the center-of-mass, and the quantity

$mR^2 = I$ is called the <u>moment of inertia</u>.

The moment of inertia is the inertia of a mass m that impedes rotational change (i.e., angular acceleration) about an axis a distance R from the mass. For a point mass, the moment of inertia is just mR^2, and for an extended object, the moment of inertia can be found by simply summing over many small masses. In this lecture, we will explicitly calculate the moment of inertia of a sphere, about an axis that passes through its center. As we need to sum over a lot of masses and a sphere is a three-dimensional object, we will need to do many integrals. Have no fear, we will break it down into many small steps.

Let's first draw the sphere and rotation axis:



Let's give the sphere a radius R and total mass M, and the sphere is being rotated about the vertical axis passing through its center.

For a small part of the sphere of mass dm, its moment of inertia dI is: ← "script r"

$$dI = \mathcal{r}^2\, dm,$$

where $\mathcal{r}$ is the distance of the little mass from the axis of rotation. Then, the total moment of inertia of the sphere is the sum over all these little moments. Taking the size of the little moments to 0, this sum is an integral:

$$I = \int dI = \int \mathcal{r}^2\, dm.$$

So, our goal is to find r and dm and do the necessary integrals.

First, as we have done with the center-of-mass, let's break apart the small mass dm into a product of a mass density ρ and small volume dV:

$dm = \rho\, dV$, where $\rho = \frac{M}{Vol}$, where Vol is

the total volume of the sphere. for a sphere of radius R, this volume is:

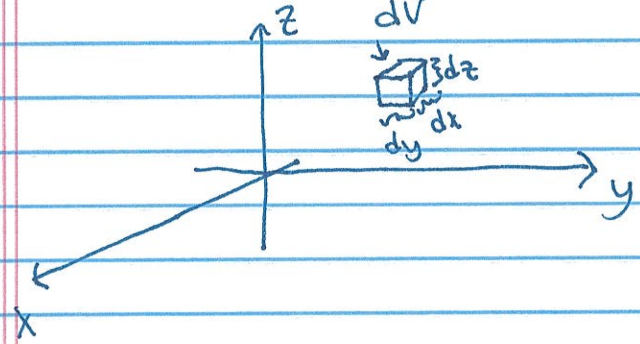
$Vol = \frac{4}{3}\pi R^3$, so the density is

$\rho = \frac{3}{4}\frac{M}{\pi R^3}$. In what follows, we will just leave

the density implicit as ρ, only plugging in the explicit expression at the end.

Now, our small moment of inertia is: $dI = r^2 \rho\, dV$, so we need to figure out the small volume dV, for some component of the sphere. To do this we need a coordinate system, just like we need coordinates to express a position vector. As always in this business, life cannot imitate art, so the value of the volume is independent of your coordinates, but you need to represent it somehow to make progress.

One possible coordinate system is Cartesian coordinates in which we represent points by their ~~horizontal~~ x (left-right), y (forward-back), and z (up-down) position. This is likely the coordinate system you are most familiar with, as we have often employed it in this class. We can draw these coordinates and a small volume in this space as:



A small volume in Cartesian coordinates is a cube of sides dx, dy, dz, so the volume of it is:

$$dV = dx\,dy\,dz$$

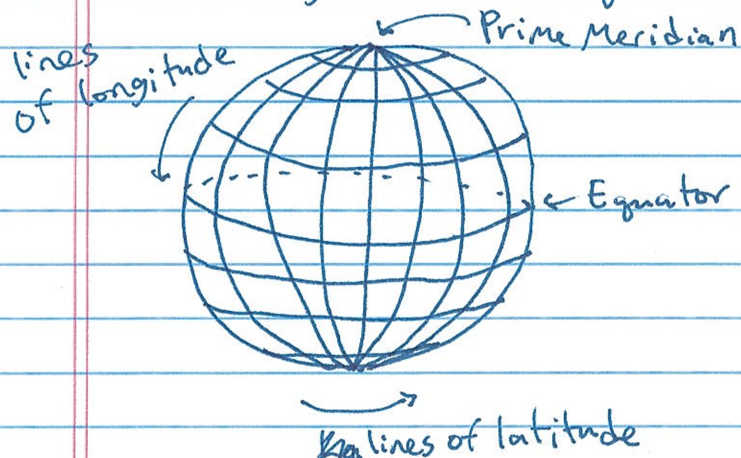
So, to calculate the volume of the sphere, for example, we just sum up a bunch of little cubes, with the constraint that the cubes form a sphere. The surface of a sphere is defined by the ~~eq~~ equation

$$x^2+y^2+z^2=R^2,$$

where R is the radius, and so points closer to the origin than the surface (the "bulk" of the sphere) are defined by the inequality:

$$x^2+y^2+z^2 \leq R^2.$$

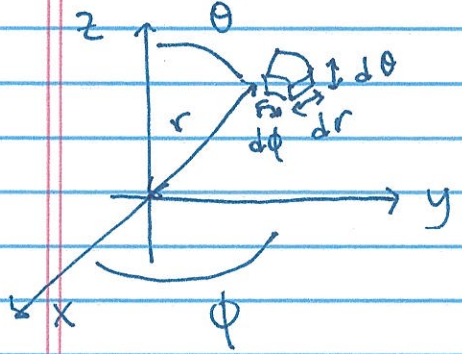
Again, I emphasize that life in physics cannot imitate art, so we could continue along this path and evaluate the moment of inertia using Cartesian coordinates. However, in practice, enforcing the relationship above is very inconvenient, because a sphere is not well approximated by a cube! So, instead of Cartesian, Let's use coordinates to express the small volume in the sphere in a way most natural for the sphere. These coordinates, or at least two of them, are familiar to you from the expression of a location on the surface of the Earth. Rather than x, y, z coordinates augmented with the constraint of being on the surface of Earth, we use latitude and longitude to express a location. That is, we put a grid on the surface of Earth in terms of relative angle from the Equator and the Prime Meridian:



Given the value of the latitude θ and longitude ϕ, we can identify a unique point on the suface of Earth. Latitude is also called the <u>polar angle</u>, because it ranges between the poles, while longitude is an <u>azimuthal angle</u> as it varies <u>about</u> the axis defined by the poles.

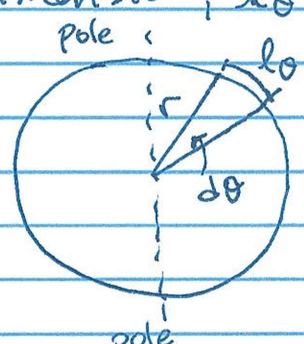
Further, for our sphere of interest that we want to calculate its moment of inertia, we need a radial coordinate r that varies from 0 (center of Earth) to R (surface of Earth). With r, θ, ϕ specified, we identify a unique point in the sphere! Also, note that restricting to the surface of Earth is very simple: we just require that $r = R$, with no squares or square-roots like in Cartesian coordinates.

With these coordinates identified, let's now figure out what the expression for the small volume dV is. ~~Well, it~~ ~~has to depend on~~ Let's draw a picture:



the volume of this little chunk is then simply the product of the length of its three sides. Note that this is not simply $dr\,d\theta\,d\phi$, because, among other issues volume has units of length-cubed, while this expression only has dimensions of length (angles are dimensionless). So we need to work a bit harder.

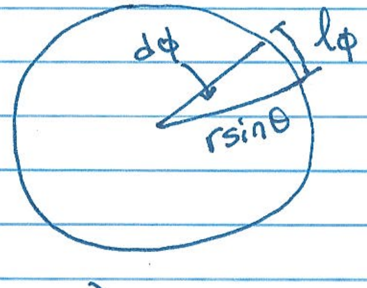
The length in the radial dimension of this chunk is indeed just dr, and to find the lengths in the θ and ϕ dimensions, we will consider projections of the sphere in different planes. Let's take a slice of the sphere along a line of longitude to determine the length in the θ dimension, l_θ:



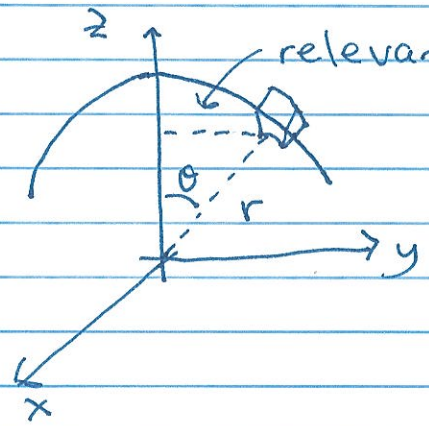
The arc length of an angular region of size $d\theta$ at radius r is simply:

$l_\theta = r\,d\theta$, which is what we need.

Now, let's take a slice along a line of latitude, an angle θ from the North pole. The picture of this for determining the length in the ϕ dimension, l_ϕ is:



Note that now the radius of this slice is $r\sin\theta$ which you can see from the geometry of where the chunk is located:



It then follows that this arclength is

$$l_\phi = r\sin\theta \, d\phi$$

Putting it all together, we find the volume of a small chunk in spherical coordinates to be:

$$dV = dr \, l_\phi \, l_\theta = r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

located a distance r from the origin and an angle θ from the North Pole. Whew!

Okay, now we just need to multiply this volume by the density, ρ, then by the distance from the axis of rotation (the z axis) and integrate, and we have the moment of inertia! We had just identified the distance from the z-axis in calculating the volume element of the chunk:

$r_z = r\sin\theta$, so then the moment of inertia of the chunk is:

$$dI = \rho r_z^2 \, dV = \rho \, r^4 \sin^3\theta \, dr \, d\theta \, d\phi$$

The total moment of inertia of the sphere is:

$$I = \int dI = \rho \int_0^R r^4 \, dr \cdot \int_0^\pi \sin^3\theta \, d\theta \cdot \int_0^{2\pi} d\phi$$

Note the simple product of one-dimensional integrals here! That will make life very simple (and something that would not have happened with Cartesian coordinates). Note also the bounds of integration: r ranges from 0 to R, the radius of the sphere; θ ranges from 0 radians (North pole) to π radians (south pole); and ϕ ranges from 0 to 2π radians (all the way around a circle. So, two of these integrals are:

$$\int_0^R r^4 \, dr = \frac{R^5}{5}, \quad \int_0^{2\pi} d\phi = 2\pi$$

The integral over θ can also be done using a u-substitution and you will derive it in homework. The answer is

$$\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$$

With $\rho = \frac{3}{4}\frac{M}{\pi R^3}$, we find the moment of inertia of the sphere to be:

$$I = \frac{3}{4}\frac{M}{\pi R^3} \cdot \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{2}{5} M R^2.$$

Whew! That's it for today!