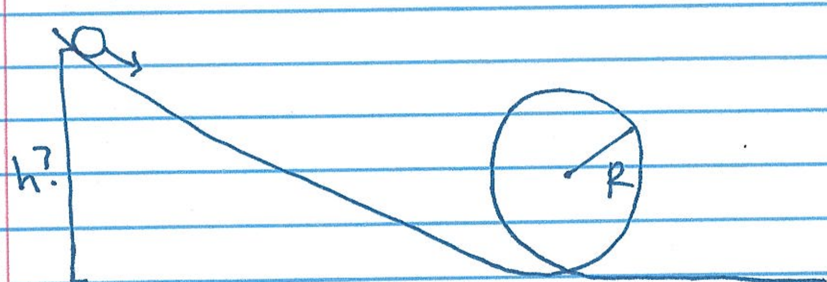
# Lecture 26 Physics 101

Welcome back! Please turn in homework!

In the previous lectures, we motivated moment of inertia as a measure of rotational inertia, analogous to mass as linear inertia that opposes changes to its motion. From one perspective, moment of inertia is just a short-hand for a complete analysis of the masses and velocities of objects undergoing rotational motion. One could just use standard Newton's laws to describe rotational motion, and everything would work out. However, we will see starting today that re-expressing Newton's laws, kinetic energies, forces, etc, in a rotational language will be extremely powerful and convenient.

In this lecture, we will revisit a problem we had analyzed with energy conservation a while ago, now accounting for rotational motion as well. We are going back to the 'ole loop-the-loop problem: given a ramp that leads to a loop-the-loop of radius R, what is the minimum height h that a ball should be released from to make it all the way around the ramp?




Let's remind ourselves of what we had done previously. First, assuming that the ball slides down the ramp without friction, we used conservation of energy to relate the initial gravitational potential energy to the energy at the top of the loop. We have:

$$mgh = \frac{1}{2}mv^2 + mg(2R),$$

where the mass of the ball is $m$ and the speed of the ball at the top of the loop is $v$. Now, for the ball to stay in the loop, it must be traveling in a circle, which enforces a minimum centripetal acceleration. At the top of the loop, centripetal acceleration is minimized if the ball just comes off the loop there, so the only force acting on it is gravity. So, demanding that the centripetal acceleration at the top of the loop is at least $g$, we find that the speed $v$ is:

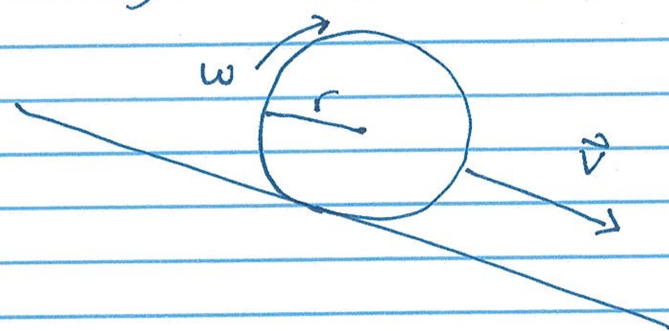
$$g = \frac{v^2}{R} \quad \text{or} \quad v^2 = Rg.$$

Inserting this into the conservation of energy from earlier, we then find

$$mgh = \frac{1}{2}m(Rg) + mg(2R) \quad \text{or that} \quad h = \frac{5}{2}R.$$

We can test this, as we did a few weeks ago, and we find the the ball doesn't make it all the way around if we release it from a height of $h = \frac{5}{2}R$. We have to release the ball higher!

So what is going on? Now, with our introduction to rotational dynamics, we know that there is more energy in the ball than just that from translation of its center-of-mass! As the ball goes down the ramp, it is both translating and rotating about its center:




Indeed, the ball is rolling without slipping, which interestingly implies that static friction is very important. The ball we roll down the ramp is a solid metal sphere (a ball bearing) so its <u>total</u> kinetic energy when rolling can be expressed as motion of its center-of-mass <u>and</u> rotation about its center-of-mass:

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v^2 + \frac{1}{2} \frac{2}{5} m r^2 \omega^2,$$

In the second equation, we inserted the expression for the moment of inertia of a sphere. Further, if the ball is rolling without slipping, we have a simple relationship between speed and angular speed:

$$v = \omega r.$$

Then, the kinetic energy of the rolling ball is

$$K = \frac{1}{2} m v^2 + \frac{1}{2} \cdot \frac{2}{5} m r^2 \omega^2 = \left(\frac{1}{2} + \frac{1}{5}\right) m v^2 = \frac{7}{10} m v^2.$$

The coefficient $\frac{7}{10}$ is kind of weird, but if it works to describe our data/demonstration, it isn't weird!

So, with this result, our conservation of energy expression becomes:

$$mgh = \frac{7}{10}mv^2 + mg(2R)$$

We still have the requirement that the minimum centripetal acceleration is $g$ so that

$g = \frac{v^2}{R}$ or $v^2 = Rg$. This then implies that

$$mgh = \frac{7}{10}mRg + mg(2R) \Rightarrow h = \left(\frac{7}{10}+2\right)R$$

or that the minimum height that the ball must be released from to go around the loop-the-loop is:

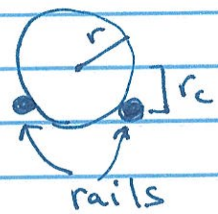
$$h = \frac{27}{10}R. > \frac{5}{2}R$$

Let's try this out!

So, we apparently need to go a little higher than even $27/10\,R$ to get the ball to go completely around the loop. Of course, we can wave our hands and say words like "sound energy losses", "friction", and the like, but there are a couple of physics points that we have ignored that I want to ask you about now.

First, the ball isn't sitting on top of the ramp;

it rides between two rails like so:




Note that the distance from the rails to the axis of rotation of the ball, $r_c$, is less than the radius of the ball, $r$.

How does this affect the minimum height that the ball needs to be released?

a) increases $h$ beyond $\frac{27}{10}R$, b) decreases $h$

c) no effect on the minimum height

Talk with your neighbors for a second!

Okay, let's figure this out. Again, the kinetic energy of the rolling ball is:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\omega^2$$

Now, however, the speed is not $r\omega = v$, but set by the smaller radius $r_c$:

$$v = r_c\omega.$$

So, the kinetic energy of the ball is:

$$K = \frac{1}{2}mv^2 + \frac{1}{5}m\frac{r^2}{r_c^2}r_c^2\omega^2 = \frac{1}{2}mv^2 + \frac{1}{5}m\frac{r^2}{r_c^2}v^2$$

$$= \left(\frac{1}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right)mv^2$$

Our conservation of energy equation is therefore:

$$mgh = \left(\frac{1}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) mv^2 + mg(2R)$$

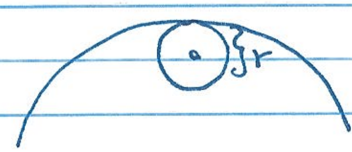
$$= \left(\frac{1}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) mgR + mg(2R),$$

where we have just inserted $v^2 = gR$ by the centripetal acceleration. Then, the minimum height $h$ is:

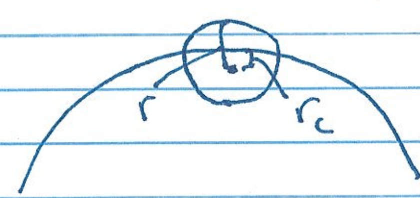
$$h = \left(\frac{5}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) R > \frac{27}{10} R,$$

because $r > r_c$. So, we have to release the ball even higher!

There's another effect we have ignored. Because of the finite size of the ball, its center-of-mass actually doesn't get to a height of $2R$ at the top of the loop. If the ball rode on top of the ramp it would be only at a height of $2R - r$:




Accounting for the separated rails that the ball rides between, this height is increased to $2R - r_c$:




How does this displacement of the ball's center-of-mass affect the minimum height further?

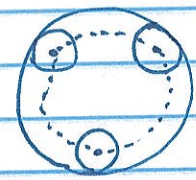
a) increases $h$ b) decreases $h$ c) no change

Talk with your neighbors for a second!

Because the center-of-mass only gets to a height of $2R - r_c$, this affects the conservation of energy expression:

$$mgh = \left(\frac{1}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) mv^2 + mg(2R - r_c)$$

Additionally, the radius of the circle through which the center-of-mass travels in the loop-the-loop is $R - r_c$, and not just $R$:



Then, the restriction on centripetal acceleration is

$$g = \frac{v^2}{R - r_c} \Rightarrow v^2 = (R - r_c)g$$

Then, conservation of energy becomes:

$$mgh = \left(\frac{1}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) mg(R - r_c) + mg(2R - r_c),$$

which slightly decreases the minimum value of $h$ by a distance $\Delta h$:

$$\Delta h = -\left(\frac{3}{2} + \frac{1}{5}\frac{r^2}{r_c^2}\right) r_c,$$ which is quite small compared to the increase in $h$ from accounting for the rails.

That's it for today! See you Friday!