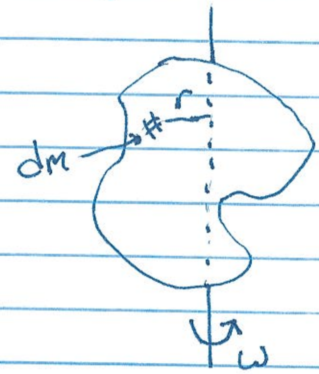
Lecture 27 Physics 101

Friday is here! Please turn in homework!

In the past week or so, we have introduced a formalism for describing the dynamics of a rotating rigid body. I want to emphasize that Newton's second law as we introduced it near the beginning of the course, as well as our expressions for kinetic energies, momenta, etc., can all be used to describe a rigid body that is rotating, but it is very convenient to rephrase expressions exclusively in terms of properties of rotation. For example if you have an object that is rotating about the axis like so:



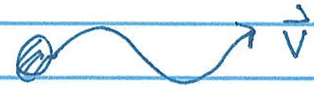
you can calculate its kinetic energy by summing up the kinetic energy of every little mass that composes it. This would be:

$$K = \frac{1}{2}\int v^2\,dm = \frac{1}{2}\int \omega^2 r^2\,dm$$

$$= \frac{1}{2}\left(\int r^2\,dm\right)\omega^2 = \frac{1}{2}I\omega^2$$

In the second equality, we note that a rigid body must rotate with the same angular velocity everywhere, and r is the distance from little mass dm to the axis of rotation. Because ω is constant over the object, it can pull out of the integral and we identify that the integral that remains is just the moment of inertia of the object. So, it becomes much more convenient to express the kinetic energy of a rotating object as $\frac{1}{2}I\omega^2$, because we only have to calculate the moment of inertia once and for all.

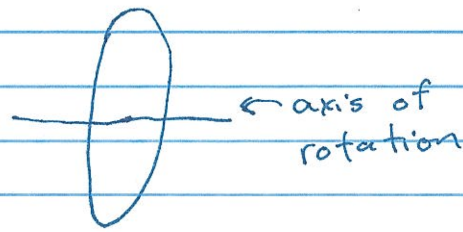
Nevertheless, there was still something slightly odd about how we denoted the angular velocity of the object. We say "angular velocity", but I have never expressed it as a vector; that is, I have written ω, not $\vec{\omega}$. This might seem a bit odd because "velocity" is definitly a vector. Further, how we have expressed the direction of rotation is a bit strange from the perspective of linear motion, for example. For an object traveling along a non-straight path, we wouldn't draw its velocity as:



yet we seem comfortable drawing the direction of rotation of an object in a non-straight manner:

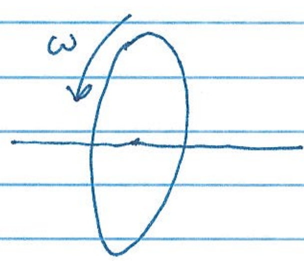
Yet another issue is that an object rotating at a constant rate should continue to rotate at a constant rate if there are no external forces acting on it. We've seen hints of this before, but the argument is simple: to rotate requires a force that keeps every point in the object in centripetal acceleration. This centripetal acceleration can be completely accounted for by internal forces in the system object; c.f. two mutually orbiting bodies interacting gravitationally. As such, there should be no ambiguity for what the angular velocity is. We shouldn't need a wonky curvy arrow to denote it. So, let's figure out a better notation.

For concreteness, let's consider rotating a bicycle tire about the axis through its center:

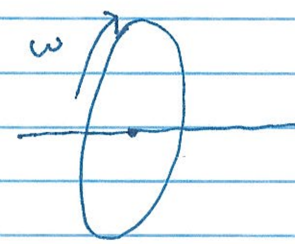


Given this axis of rotation, how many options are there for the direction of rotation?

This is amazingly simple and profound: given an axis of rotation, there are only <u>two</u> options for the direction of rotation. The wheel can rotate with the front of the tire moving <u>down</u> or <u>up</u> from your perspective:

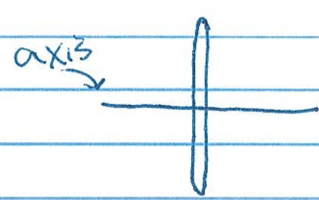


or

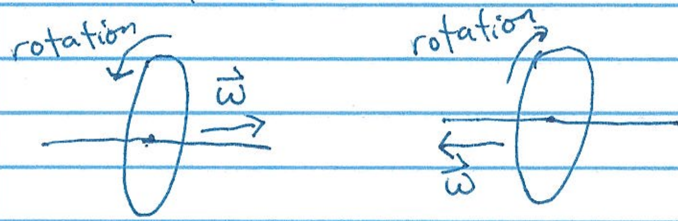


However "moving up" or "moving down" is not a unique way to define rotation. If the wheel looks to be moving up to you, what does it look to someone on the other side of the wheel? They see it moving down! So we need another way to denote the direction of rotation.

Let's look at the wheel head-on:

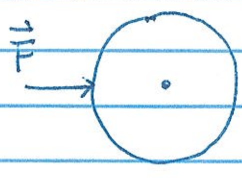


Note that there are the same numbers of direction of rotation (2) as sides of the wheel! Another way to say this is that, given the axis of rotation, we can move along it to the left or to the right. So, a natural way to express the angular velocity is as a vector that points either to the left or right along the axis. Then, the axis is clear and the direction of rotation is unambiguous, given a convention for mapping direction of rotation to a direction along the axis. This mapping is called the right-hand rule. What one does is curl the fingers on your right hand in the direction of rotation and then your thumb points in the direction of the angular velocity vector, $\vec{\omega}$. For the wheel we have:



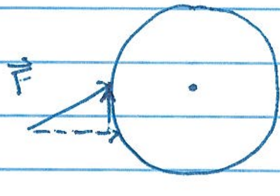
This is one of many right-hand rules you will learn in physics this year.

The second right-hand rule, we will introduce now. How do get the wheel rotating in the first place? Just like with getting an object to move from rest, we have to apply a force. Unlike for linear motion, however, just any old force won't do. For example, if I pushed the wheel perpendicular to it (radially), in the direction of the center of the wheel, would the wheel rotate? I only care if it rotates, not if it moves otherwise. That is, I apply a force:

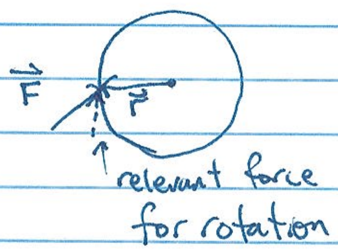


No! To convince yourself of this, try to close a door by pushing on the edge of the door, toward the hinge.

To get the wheel rotating, we have to apply a force with some non-zero component tangent to the surface of the wheel:



The component of the force in the direction of the axis of rotation is doing nothing to help actually rotate the wheel. So, the component of the force that works to rotate the wheel is perpendicular to the position vector at which the force is applied:



Further, more change in rotation is accomplished by pushing farther from the axis of rotation. If we push a door right at the hinge it is very hard to close the door, which is why door handles are at the outside edge of the door. Similarly, for the same force, there is much less change in rotation of the wheel if the force is applied near the axis of rotation. By the way, does this imply that higher or lower gears on a bike correspond to a force applied farther from the axis/axle?

So, if we only want the component perpendicular to the position with respect to the axis and we want to be further away from the axis to rotate more easily, this suggests that the "rotational force" or torque has magnitude:

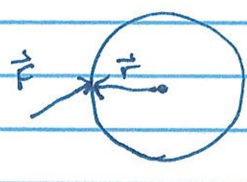
$\tau = Fr\sin\theta$, where θ is the angle between $\vec{F}$ and $\vec{r}$.

This torque is the agent that enacts rotational change, so, by Newton's second law, is equal to the product of the rotational inertia and angular acceleration:

$\tau = I\alpha$, where I is the moment of inertia (rotational inertia) and α is the angular acceleration.

This is referred to as Newton's second law for rotations. For forces acting on two masses connected by a rigid rod, we had derived this expression simply through manipulations of Newton's second law for linear motion.

This is great, but it's not a vector equation yet. We need to figure out the direction of angular acceleration that a given torque induces. Let's consider a force on the tire as such:



What direction will the wheel rotate? So therefore what is the direction of angular velocity?

a) no rotation b) $\vec{\omega}$ ⊗ (in to page) c) $\vec{\omega}$ ⊙ (out of page)

Talk amongst your neighbors for a second!

Let's place the force and position vectors with their tails together:

This force will enact a rotation clockwise from our perspective, so angular velocity/acceleration are both <u>in to the page</u>. We used the right-hand rule to find that direction of angular velocity; can we construct a right-hand rule with $\vec{r}$ and $\vec{F}$ that produces the same direction? Indeed, and the right-hand rule for the direction of torque is:

1) Point fingers in direction of position vector $\vec{r}$
2) Curl fingers in direction of force vector $\vec{F}$
3) Thumb points in direction of torque.

Does this work for our example here? Remember, it's called the right-hand rule for a reason: you have to use your right hand! This isn't some conspiracy against southpaws, we just need some convention for defining the direction of angular vectors.

To end today, I want to express the Newton's second law for rotations in vector form. It should encode the direction information we have discussed and this can be accomplished by a vector cross product:

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$$

The magnitude of the cross product of two vectors is:

$$|\vec{r} \times \vec{F}| = rF\sin\theta,$$ where θ is their relative angle.

You'll explore the cross product in homework.

Have a good weekend!