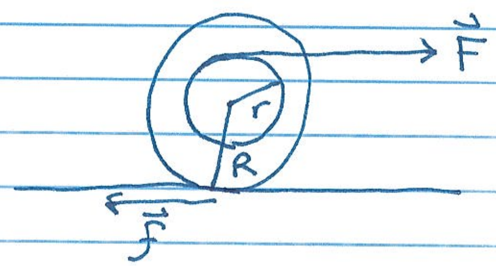
Lecture 29 Physics 101

Welcome to Wednesday! Please turn in homework and remember there is an exam on Friday!

In this lecture, the last before the exam, we're going to have some fun with demos regarding the application of torque to understanding the connection between rotational and translational motion. To start today, let's analyze pulling a rope connected to a spool along the ground with a force $\vec{F}$, without ~~sp~~ slipping. The picture of this is:

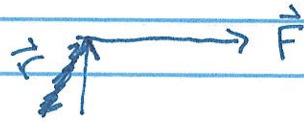


The radius of the spool is R, while the radius of the inner section where the rope is wrapped around is $r<R$. I have denoted the pull force $\vec{F}$ and the force of friction $\vec{f}$, responsible for the rolling without slipping. Forces that act in the vertical direction (normal force and gravity) will be irrelevant to this discussion, so we ignore them. With this setup, let's begin.

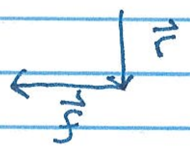
First, let's find the net force on the spool, so find the acceleration of the center-of-mass of the spool. We have

$\vec{F}+\vec{f}=m\vec{a}$, where m is the mass of the spool.

In components, this is just: $F-f=ma$, along the horizontal axis. Now, let's identify the torques on the spool. The torques from both $\vec{f}$ and $\vec{F}$ act in the same direction:



$$\vec{r} \times \vec{F} = \vec{\tau}_F \otimes$$



$$\vec{r} \times \vec{f} = \vec{\tau}_f \otimes$$

by the right hand rule. Recall that the right hand rule is:

1) Point fingers on right hand in direction of $\vec{r}$, the vector that stretches from the axis of rotation to the point of application of the force

2) Curl fingers in direction of applied force.

3) Thumb points in direction of torque.

Additionally, the angle between the vectors $\vec{r}$ and $\vec{F}$ (or $\vec{f}$) are both $\theta = 90°$, so the magnitudes of the torques are:

$$\tau_F = rF \ , \ \tau_f = Rf$$

Then, Newton's second law for the rotation of the spool is:

$$\tau_F + \tau_f = I\alpha \Rightarrow rF + Rf = I\alpha,$$

where I is the moment of inertia of the spool and α is its angular acceleration. Now, if the spool rolls without slipping on the ground, this relates linear and angular accelerations a and α:

$$\alpha = \frac{a}{R},$$ so that the torque equation is

$$rF + Rf = I\frac{a}{R}.$$

Now, we have two equations (net force and net torque) and two unknowns (a and f). Isolating f in the net force equation:

$$f = F - ma,$$

and plugging it into the net torque equation, we have:

$$rF + R(F - ma) = \frac{I}{R}a \Rightarrow (r+R)F = \left(mR + \frac{I}{R}\right)a,$$

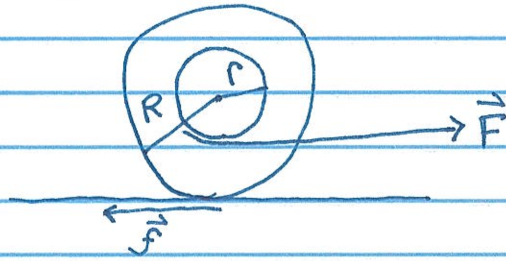
or the acceleration of the spool is:

$$a = \frac{r+R}{mR + \frac{I}{R}}F,$$ to the right.

Not surprisingly, if the mass or moment of inertia of the spool increases, this acceleration decreases, for the same force. We won't worry about solving for the moment of inertia here.

Let's see this in action! Does the spool accelerate right, as expected?

Now, I want to do something radical: what happens if I pull the spool as before, but now flip it over? That is, the set up now is:



Does the spool:

a) accelerate right b) accelerate left or

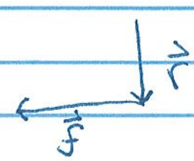
c) no acceleration

Talk with neighbors for a second!

Well, let's analyze this system, identically to what we did before. The net force equation is:

$\vec{F} + \vec{f} = m\vec{a}$, or in components, $F - f = ma$.

The torque of the friction is identical to earlier:



$\vec{r} \times \vec{f} = \vec{\tau}_f \otimes$

Now, the torque from the force $\vec{F}$ is in the opposite direction, by the right-hand rule:

$\vec{r} \times \vec{F} = \vec{\tau}_F \odot$

So now, Newton's second law for rotations is:

$$\vec{\tau}_f + \vec{\tau}_F = I\vec{\alpha} \Rightarrow \tau_f - \tau_F = Rf - rF = I\alpha.$$

As before, we eliminate f from the net force and relate angular and linear accelerations with a radius factor:

$$R(F - ma) - rF = \frac{I}{R}a.$$

Rearranging and solving for a, we find:

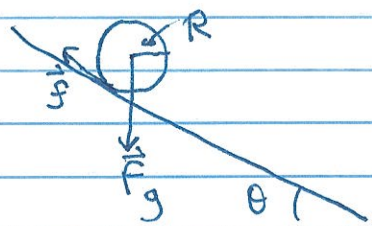
$$(R-r)F = \left(mR + \frac{I}{R}\right)a \quad \text{or that}$$

$$a = \frac{R-r}{mR + \frac{I}{R}}F.$$

Because $R > r$, this is still positive, so the spool <u>still</u> accelerates right. However, its magnitude is less than earlier because of the subtracted factor in the numerator.

Let's try this out!

Finally, I want to analyze the dynamics of an object rolling down a ramp:



Here, I have drawn the relevant forces for acceleration down the ramp (no normal force), and the angle of the ramp is θ, while the radius of the object is R. We will also assume that the mass and moments of inertia of the object are m and I, respectively.

As before, let's analyze the relevant forces that accelerate the object down the ramp:

$$\Sigma \vec{F}_x = m\vec{a}_x \Rightarrow F_g \sin\theta - f = mg\sin\theta - f = ma,$$

where a is the acceleration of the center-of-mass of the object.

Now, let's analyze torques for rotation about the center of mass of the object. Gravity has no torque because it acts at the center-of-mass; its lever arm is 0. Further, though not illustrated, normal force also exerts no torque because it acts at the surface of the object, in the direction of its axis of rotation. The only force that exerts a torque is friction, so

$$\tau_f = I\alpha \Rightarrow fR = I\alpha = I\frac{a}{R},$$

where we replaced angular acceleration α with linear acceleration via

$$a = R\alpha,$$

because the object does not slip.

Solving for friction f in the torque equation, we find

$f = \frac{I}{R^2}a$, and inserting it into the force equation we have:

$mg\sin\theta - \frac{I}{R^2}a = ma$, or, solving for acceleration a,

$$a = \frac{mg\sin\theta}{m + \frac{I}{R^2}} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}.$$

Thus, we immediately see that objects that roll without slipping accelerate down a ramp more slowly if they have a larger moment of inertia, I. However, two objects, identical in shape, but that differ in size and mass, accelerate down the ramp <u>identically</u>.

Let's test this out!

That's it for today! See you Friday for the exam!