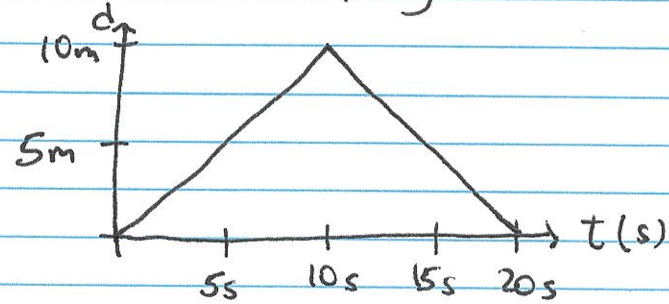
Physics 101 Lecture 3

This lecture we are continuing our study of the description of motion in one-dimension. We've discussed displacement (that is, a measure of position from a specified origin), and we'll continue our study today with velocity and acceleration.

Last lecture, we had drawn the position vs. time graph of me walking across the well down here. Recall it looked something like:



The story this graph tells is that I walked away from the origin for 10 seconds traveling 10 m and then returned to the origin, which took another 10 seconds. Not only does this graph tell the story of my position as a function of time, but it also tells the story of the change in my position as a function of time. That is, from one moment in time to the next, this graph encodes the rate at which my position changed. Whenever you hear the word "rate" you should think "slope", so we can identify the slope at any point in time to study the rate of change of position.

For a function $d(t)$, the displacement as a function of time, the slope between two points at time $t+\Delta t$ and t is:

$$\text{Slope} = \frac{d(t+\Delta t)-d(t)}{\Delta t}$$

Strictly speaking a slope only makes sense for a line; however, we can imagine taking Δt as small as possible to determine the slope at two neighboring times, infinitesimally close to one another. This limiting procedure produces a derivative:

$$\lim_{\Delta t \to 0} \frac{d(t+\Delta t) - d(t)}{\Delta t} \equiv \frac{d}{dt} d(t).$$

We call the time derivative of displacement the <u>velocity</u> $v(t)$, which is the instantaneous change in displacement.

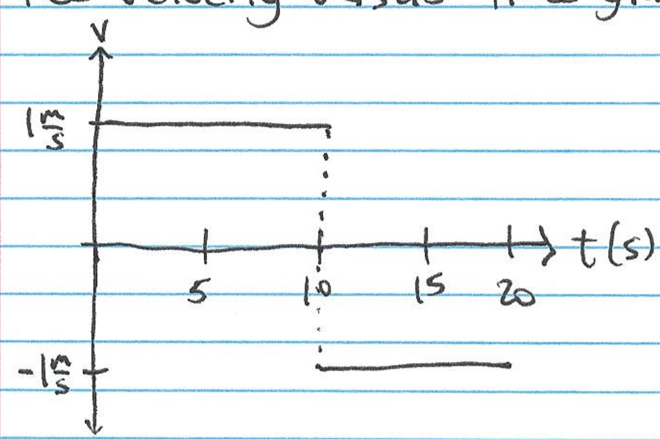
For the position versus time graph we're studying, we can produce the velocity vs. time graph straightaway. In the first 10s, I traveled 10 m, so the velocity is

$$v(t<10s) = \frac{10m}{10s} = 1\,\frac{m}{s}.$$

In the next 10 s, I traveled −10 m (I went left instead of right) so the velocity is

$$v(10s<t<20s) = \frac{-10m}{10s} = -1\,\frac{m}{s}.$$

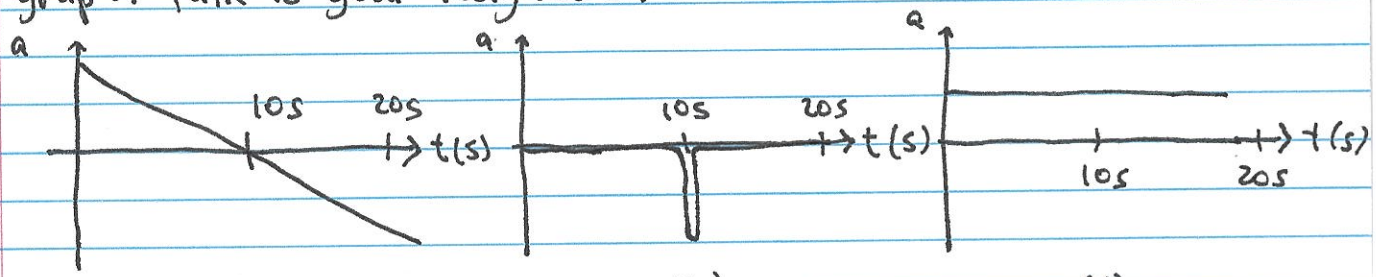
The velocity versus time graph is thus:



Why stop at velocity? We can also study the rate of change of velocity, called the <u>acceleration</u> $a(t)$:

$$a(t) \equiv \frac{d}{dt} v(t) = \frac{d^2}{dt^2} d(t).$$

As acceleration is the time derivative of velocity, which is itself the time derivative of displacement, acceleration is the second time derivative of displacement. Can we determine the acceleration versus time graph for my path? Below are three possible plots. Which one do you think is correct, based on the velocity vs. time graph? Talk to your neighbors!



(a) (b) (c)

I've left off labels/ticks on the ordinate (y-axis). What do you think? Note that for $t < 10s$, the velocity is constant; it does not change in time. Therefore

$$a(t < 10s) = 0$$

Further, for $t > 10s$, the velocity is also constant, though perhaps a different value than the velocity before 10s. Nevertheless, the acceleration is the instantaneous change in velocity and so too is

$$a(10s < t < 20s) = 0$$

Around 10s, however, something interesting happens: my velocity rather rapidly changes from positive to negative. Thus, around the instant of 10s, I experience a large negative acceleration. Why negative? Because my velocity just after 10s is smaller than my velocity just before. These considerations imply that (b) is the correct graph.

So, we can describe motion in different capacities by taking more derivatives:

$$\begin{tikzcd}[column sep=large]
\text{displacement} \arrow[r, "\frac{d}{dt}"] \arrow[rr, bend right, "\frac{d^2}{dt^2}"'] \arrow[rrr, bend left, "\frac{d^3}{dt^3}"] & \text{velocity} \arrow[r, "\frac{d}{dt}"] \arrow[rr, bend right, "\frac{d^2}{dt^2}"'] & \text{acceleration} \arrow[r, "\frac{d}{dt}"] & \text{jerk} \arrow[r] & {}
\end{tikzcd}$$

etc. In a couple weeks, we'll make this more precise, but it's good to think about it now. Here's a question: can you "feel" velocity or acceleration? Imagine you are on a train traveling at constant speed on very smooth rails. If you never looked out a window, could you tell that you were actually moving at all? Now, instead imagine that the conductor slammed on the brakes, violently changing the velocity of the train. Without looking out a window could you tell that the train was stopping?

This exercise manifests a few things. First, this is refferred to as a "gedankenexperiment" or thought experiment in German. Thought experiments, in which we use our experience to test our physical intuition, is an extremely powerful tool for making sense of Nature. Second, the fact that you apparently could not tell that the train was moving at a constant velocity but could tell that it was accelerating suggest a symmetry of Nature. We could all be traveling at a constant rate, and there is no experiment we could do to determine if it were so. By Noether's theorem, this constant velocity or "boost" symmetry suggests there is a conservation law. What do you think that might be? Why we feel accelerations and not velocities is encapsulated in Newton's laws, which we will discuss in the coming weeks.

For now, let's just study a system which undergoes constant acceleration (note that 0 is also constant). Let's call this constant a, which is some number. As acceleration is the rate of change in time of the rate of change in time of displacement, the dimensions/units of a are meters per second per second (m/s^2). ~~Velocity~~ ~~is~~ As acceleration is the rate of change of velocity and is constant, velocity must be <u>linear</u> in time. In general, we can then write:

$$v(t) = at + v_0,$$

where $v_0 = v(t=0)$. Note that this expression is <u>only</u> true for <u>constant</u> acceleration a. What is the displacement with constant velocity? We have the relationship

$$v(t) = \frac{d}{dt} d(t), \text{ for displacement } d(t).$$

To solve for $d(t)$, we need to "undo" the derivative, or anti-differentiate. By the Fundamental Theorem of Calculus, the anti-derivative is just the <u>integral</u>. So, given the velocity $v(t)$, we just integrate to find $d(t)$.

Doing this, we find:

$$d(t) = \frac{1}{2}at^2 + v_0 t + d_0,$$

where $d_0 = d(t=0)$. Let's now differentiate to see if this makes sense. Recall that the derivative of a power is:

$$\frac{d}{dt} t^n = n t^{n-1}, \text{ for some } n. \text{ Then,}$$

$$\frac{d}{dt} d(t) = \frac{2}{2} at + v_0 + 0 \cdot d_0 = at + v_0 = v(t), \text{ as expected.}$$

If you find yourself struggling with these manipulations, please come see me in office hours.

For the remainder of this lecture, we are going to use these results to measure the height of the ceiling of this room. What we will do is the following. We will throw a ball up to the ceiling, just to the point of touching the ceiling. Because the ball has an initial, non-zero velocity, and stops moving up at some point, the ball is accelerating. This accelerating is due to the gravitational pull of the Earth on the ball and near the surface of the Earth is approximately constant with the value:

$$a_g \equiv g = 9.8\ m/s^2.$$

Note as the ball moves <u>up</u>, its velocity decreases, therefore it experiences <u>negative</u> acceleration. Then, the displacement of the ball from the floor can be expressed as:

$$h(t) = -\frac{g}{2}t^2 + v_0 t + h_0.$$

h_0 is the initial height of your hand right when you throw the ball. To know the ceiling height, we apparently need to know the time t at which the ball touches the ceiling. What else happens at that time? The ball's velocity immediatly before was moving upward (positive) while immediately after is moving downward (negative). So what must the velocity be at the moment it touches the ceiling, its highest point? Zero!

With the expression for velocity: $v(t) = -gt + v_0$,

we know that at time $t = T_{ceiling}$, the velocity is 0:

$$v(t = T_{ceiling}) = 0 = -g T_{ceiling} + v_0,$$ or, solving for the

initial velocity v_0, we find:

$$v_0 = g T_{ceiling}.$$

Now, the height at $t = T_{ceiling}$ of the ball is just the ceiling height, so we can express the ceiling height as:

$$h_{ceiling} = h(t = T_{ceiling}) = -\frac{1}{2} g T_{ceiling}^2 + g T_{ceiling}^2 + h_0$$

$$= \frac{1}{2} g T_{ceiling}^2 + h_0.$$

So, all we need to measure the ceiling height is to measure the time it takes for the ball to reach the ceiling, $T_{ceiling}$!

Let's do this! I need two volunteers: one to throw the ball and the other to time it.

Don't forget labs and conferences start this week! Also, there's homework for Wednesday!