# Lecture 30 Physics 101

Welcome back from the weekend to this short week! Please turn in homework. Reminder that there are no labs or conferences this week. Also, homework assigned today is due Wednesday, Nowember 27. Homework "assigned" on Wednesday isn't due until Wednesday, December 4, after we return from Thanksgiving break.

For today and Wednesday, we are going to wrap up our discussion of rotational dynamics with a discussion of the final space-time conservation law. Let's start by reminding ourselves about the Newton's second law for rotational motion:

$$\vec{\tau}_{net} = I\vec{\alpha},$$

where $\vec{\tau}_{net}$ is the net torque about a defined axis, $I$ is the moment of inertia of the system about that axis, and $\vec{\alpha}$ is the angular acceleration. As an angular acceleration, $\vec{\alpha}$ is the time derivative of the angular velocity:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

If the moment of inertia is constant, $\frac{dI}{dt}=0$, we can re-write this Newton's second law as:

$$\vec{\tau}_{net} = I\frac{d\vec{\omega}}{dt} = \frac{d}{dt}(I\vec{\omega})$$

In this form, it is very similar to Newton's law for linear motion:

$$\vec{F}_{net} = \frac{d}{dt}(m\vec{v}),$$

where we called $m\vec{v} \equiv \vec{p}$, the (linear) momentum.
For Newton's second law for rotational motion, we instead refer to $I\vec{\omega}$ as the <u>angular momentum</u>:

$$\vec{L} \equiv I\vec{\omega},$$

and so Newton's second law for rotations can equivalently be written as:

$$\vec{\tau}_{net} = \frac{d}{dt}\vec{L}.$$

That is, changes in angular momentum are enacted by external torques (just like forces affect linear momentum).

Note that, by Newton's third law, the torques <u>internal</u> to a system always cancel pairwise, just like we observed with linear momentum. So, if there are no external torques, then the time derivative of angular momentum is 0:

$$\frac{d}{dt}\vec{L} = 0, \text{ if } \vec{\tau}_{net} = \vec{\tau}_{ext} = 0.$$

That is, angular momentum is <u>conserved</u> if there are no external torques.

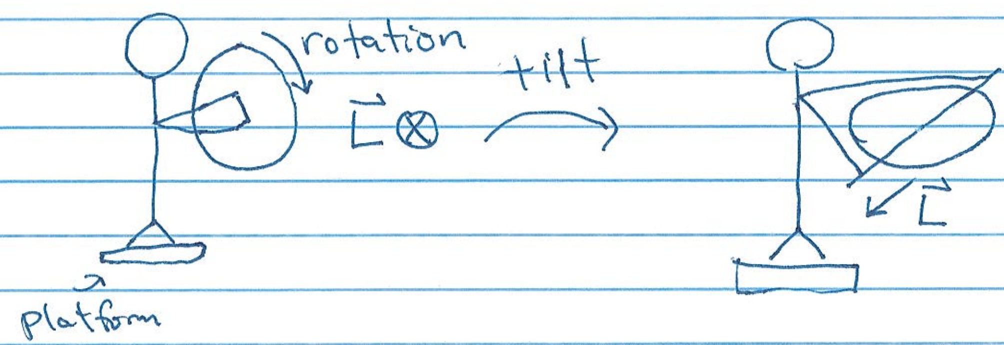
Uh oh, we have a new conservation law, so you know that that means a digression into its consequences by Noether's theorem. Angular momentum conservation means that rotation about (at least) one axis is unchanged in time, defined by $\vec{L} = I\vec{\omega}$. If a system has non-zero angular momentum, then that system sweeps through arbitrary angles. Equivalently, the orientation of the system continually changes. If angular momentum is conserved, then there is no special angle or orientation; all orientations are equivalent and exhibit the same laws of physics. This is just to say that conservation of angular momentum means that all orientations are equivalent, or that there exists a symmetry of spatial rotation of our system.

So, we have identified the following symmetries and conservation laws in our study this semester:

| Symmetry | Conservation Law |
|---|---|
| Time Translation | Energy |
| Spatial Translation | Momentum |
| Spatial Rotation | Angular Momentum |

We believe that in our universe, energy, momentum, and angular momentum are all conserved. So, with that assumption, what does our universe look like if it is unchanged by the actions of time translation, spatial translation, and/or spatial rotation? I'll leave that for you to think about.

Coming back from esoteria, let's see if we can understand some consequences of angular momentum conservation. As a concrete example, I'm going to perform the following demo. I am first going to get our old friend the bicycle wheel spinning pretty fast. Then, I will stand on this platform that can spin and Owen will hand me the wheel. Now with the wheel in hand, I will tilt the wheel, so it makes a non-right angle with respect to the ground. A picture of this is:




What happens?

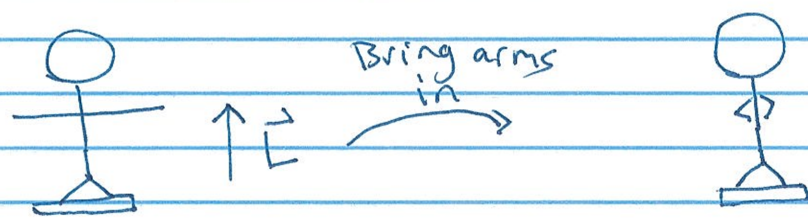
a) Nothing    b) I start rotating with $\vec{\omega}\uparrow$

c) I start rotating with $\vec{\omega}\downarrow$

Talk with your neighbors for a second!

Initially, the direction of angular momentum was into the page, ⊗ $\vec{L}$. After I tilted the wheel, its angular momentum picked up a component in the downward vertical direction. However, my tilting the wheel was an internal torque to the system of the wheel and me, and there were no relevant external torques. Therefore ~~the~~ angular momentum will be conserved. That is, the direction of the combined angular momentum of me and the wheel must still point into the page. To accomplish this, I must have an angular momentum that points upward, to cancel the change in angular momentum of the wheel. Let's test this out!

That's not the only way to affect angular momentum. Let's consider another demonstration; this fine where I am rotating standing on the platform with my arms outstretched. Then, I will bring my arms in close to my body. An illustration of this is




What happens when I do this?

a) My angular velocity increases b) angular velocity decreases

c) angular velocity stays the same

Talk to your neighbors for a second!

As with tilting the wheel, bringing my arms in is a completely internal action to the system of, well, me, so it does not affect my angular momentum. Therefore, my angular momentum before and after moving my arms is unchanged. However, with my arms out stretched, I have mass far from the axis of rotation and so by bringing them close to my body, I can decrease my moment of inertia significantly. So, if my initial moment of inertia is larger than my final moment of inertia: $I_i > I_f$ and angular momentum is unchanged:

$$\vec{L}_i = \vec{L}_f \Rightarrow I_i \vec{\omega}_i = I_f \vec{\omega}_f,$$

my angular velocity must <u>increase</u> for this equation to hold. Let's test it out!

We'll play around with some more, crazier predictions from $\vec{\tau} = \frac{d}{dt}\vec{L}$ on Wednesday. For the rest of this lecture, I want to introduce another, equivalent, definition of angular momentum.

Let's go back to the definition of torque:

$\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the <u>lever-arm</u>, the position vector that stretches from the axis of rotation to the point at which the force $\vec{F}$ is applied. Now, using linear Newton's second law, we can replace $\vec{F}$ with $\frac{d\vec{p}}{dt}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

If the lever arm vector $\vec{r}$ is constant in time, $d\vec{r}/dt = 0$, we can move the derivative all the way to the left:

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

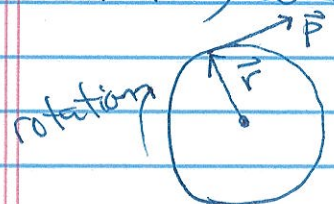
We had already established that torque is equal to the time derivative of angular momentum:

$$\vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}\times\vec{p}),$$

Therefore, we have another definition of angular momentum:

$$\vec{L} = \vec{r}\times\vec{p}.$$

Let's see if this makes sense for our rotating wheel:

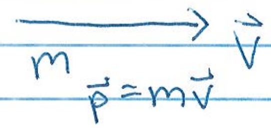



$\vec{\omega} \otimes$ $\qquad$ $\vec{r}\times\vec{p} \otimes$

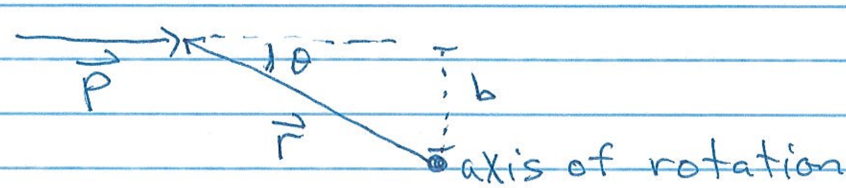
$I\vec{\omega} = \vec{L} \otimes$

What about $\vec{r}\times\vec{p}$? Again, $\vec{r}$ is the vector from the axis of rotation to a point on the wheel that is rotating. The momentum of such a point on the wheel is in the direction tangent to the wheel. A point on the wheel is moving in a circle, and the direction of velocity/momentum of circular motion is tangent to the circle. So, to determine the direction of $\vec{r}\times\vec{p}$, we use the right-hand rule: point our fingers in the direction of $\vec{r}$, curl in the direction of $\vec{p}$, and thumb points in direction of $\vec{r}\times\vec{p}=\vec{L}$. This is exactly what we get with $\vec{L}=I\vec{\omega}$ and the right-hand rule for angular velocity!

One final note: this definition of angular momentum as $\vec{r}\times\vec{p}$ enables us to define angular momentum even for linear motion of a particle. Let's imagine that a particle of mass $m$ with velocity $\vec{v}$ is passing by:




There are no external forces on the particle, so it travels in a straight line. Now, we can just pick some arbitrary point in space and call it the "axis of rotation":




What is the angular momentum about this axis of rotation? First the direction is

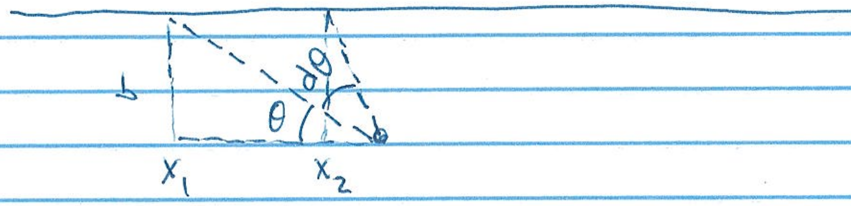
$$\vec{L} = \vec{r} \times \vec{p} \;\otimes\text{, by the right-hand rule.}$$

Next, its magnitude is

$$|\vec{L}| = |\vec{r} \times \vec{p}| = rp\sin\theta = pb.$$

The perpendicular distance b is called the "impact parameter", and is the ~~point~~ distance of closest approach of the particle to the axis of rotation. Because there are no external forces, the impact parameter is constant for a given trajectory, and so too is angular momentum constant as there are no net torques.

As to why linear motion can correspond to non-zero angular momentum, note that as the particle travels, it sweeps out angles with respect to the axis of rotation over time:




$$x_2 - x_1 = \frac{b}{\tan(\theta + d\theta)} - \frac{b}{\tan\theta}$$

$$\cong \frac{b}{\tan\theta}\left[\frac{1}{1+\frac{d\theta}{\cos\theta\sin\theta}} - 1\right] \cong -\frac{b\,d\theta}{\sin^2\theta}$$

Note that it takes time $v\,dt = x_2 - x_1 = \frac{p}{m}dt$ to travel that distance. Equating these, we have

$$\frac{b\,d\theta}{\sin^2\theta} = \frac{p}{m}dt \Rightarrow \frac{mb^2}{\sin^2\theta}\frac{d\theta}{dt} = pb = L$$

where we note that $\frac{d\theta}{dt} = \omega$ and the distance $r$ from the axis of rotation to the particle's location is

$r = \frac{b}{\sin\theta}$, so the moment of inertia is:

$$I = mr^2 = m\frac{b^2}{\sin^2\theta},$$ as expected!