Lecture 31 Physics 101

Welcome to the final lecture of this short week! Please turn in homework, and remember that the homework assigned today is due next Wednesday, December 4.

Last lecture, we had rewritten Newton's second law for rotations in terms of the change imparted on angular momentum $\vec{L}$:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}, \quad \text{where } \vec{L} = I\vec{\omega} = \vec{r} \times \vec{p},$$

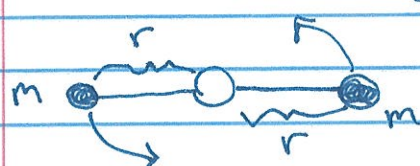
the product of moment of inertia and angular velocity or the cross product of the position vector $\vec{r}$ from the axis of rotation to the location of the particle which carries momentum $\vec{p}$. If there are no relevant external torques, $\vec{\tau}_{net} = 0$, and so angular momentum is conserved, or unchanging, in time. By Noether's theorem, angular momentum conservation means that the laws of physics are independent of orientation about the identified axis of rotation. In our universe, we believe that angular momentum is conserved so this orientation-independence has profound consequences on the structure of the universe.

In this lecture, we will relax the assumption of $\vec{\tau}_{net} = 0$ and attempt to understand the physics of a non-trivial Newton's second law for rotations. The first thing we will do is to address the lecture ticket from last homework. The question there was let's say that I am rotating on the non-OSHA approved rotating platform with my arms outstretched, holding 2 Kg weights in each hand. While rotating like this, I release the weights. What happens? Does my rotation rate

a) increase b) decrease c) stay the same

What do you think? Let's test it out!

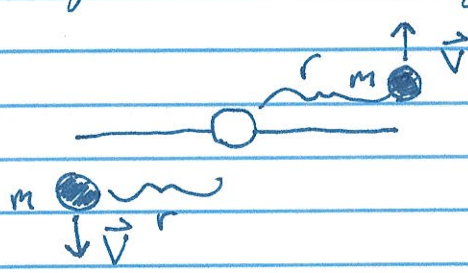
To analyze this problem, the fact that the weights fall due to gravity is completely a red-herring and doesn't affect the result. So let's instead imaging that we are just rotating in space, fare from any stars. Before releasing the weights, I (and the weights) am rotating as:



$\vec{\omega} \odot$ and our angular momentum is

$$L_{tot} = I_{me}\,\omega + 2mr^2\omega$$

When I release the weights, they cease their circular motion and travel in a straight line tangent to the original circular motion:



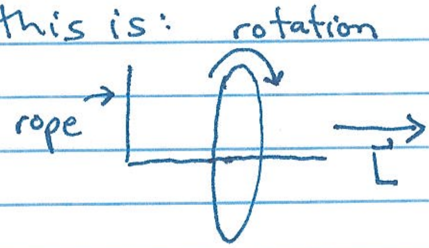
Note that the speed v is just

$v = r\omega$ so that the angular momentum of the two masses with respect to my head (axis of rotation) is

$$L_{masses} = 2|\vec{r}\times\vec{p}\,| = 2r\,(mv) = 2r\,(mr\omega)$$
$$= 2mr^2\omega$$

This is identical to before releasing the masses! Simply releasing the ~~masse~~ masses exerted no torque so necessarily angular momentum is conserved in this reaction. Therefore if angular momentum is conserved and the angular momentum of the masses didn't change, then so too must my angular momentum remain the same!

Okay, good enough, now let's analyze another rotating system. We're going to bring back our bicycle wheel and get it rotating. Once rotating, I'm going to only hold it by a rope attached to the end of one of the handles. A picture of this is:



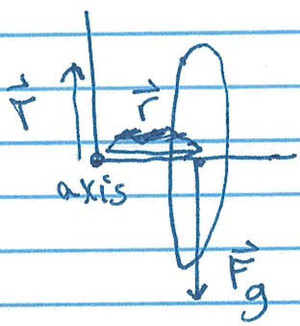
My question to you is: what happens? What does the wheel do once this system is released to the world? Possible ~~answers~~ answers include:

a) Nothing, remains stationary b) falls to vertical position

c) revolves about rope d) oscillates like a pendulum about end of rope

Talk amongst your neighbors for a bit!

To analyze this system, let's first consider what happens if the wheel were stationary and not rotating. In that case, we will analyze the torques of the system, about an appropriate axis. Note that the end of the rope is stationary in this setup, so we will call the end of the rope the axis of rotation. Then, let's draw the forces on the wheel:

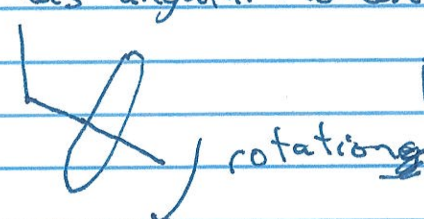


There is a tension in the rope and gravity acts at the center-of-mass of the wheel. The tension exerts no torque because it is applied to the axis of rotation. By contrast, gravity does exert a torque about this axis because it is displaced from the axis and an angle of $90°$ with respect to $\vec{r}$. By the right-hand-rule, the direction of this torque is

$\vec{\tau}_g = \vec{r} \times \vec{F}_g \otimes$ while the magnitude is

$|\vec{\tau}_g| = r F_g = rmg$, which is indeed non-zero.

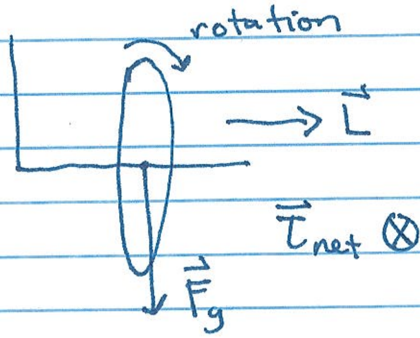
Of course, we know what happens when we release the wheel in this case: it falls, swinging clockwise, and so has angular momentum into the page:



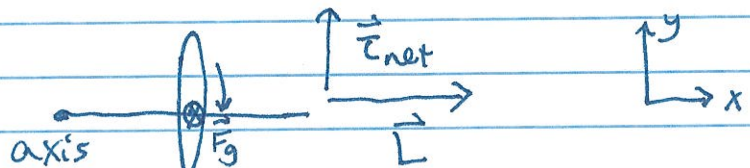
$\vec{L} = \otimes$

This is exactly as predicted by $\vec{\tau} = \frac{d\vec{L}}{dt}$.

Okay, that was easy. Now let's get the wheel rotating and think about what happens. We have the same torque as in the non-rotating case:



But we have a non-zero angular momentum, and that makes all the difference. Let's first consider what happens a <u>very</u> short time after we release the system to the wild. To do this, let's draw an overhead picture of the system which will clarify what's going on:



For arguments sake, let's call the initial angular momentum

$$\vec{L}_i = L\hat{x} = L\hat{\imath},$$ and the direction of torque

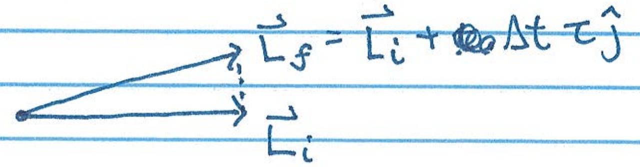
$$\vec{\tau}_{net} = \tau\hat{y} = \tau\hat{\jmath}.$$

Then, by Newton's second law for rotations:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \tau\hat{\jmath} \cong \frac{\vec{L}_f - \vec{L}_i}{\Delta t}$$ where $\vec{L}_f$ is the angular momentum evaluated a <u>time</u> Δt after I release the system. Solving for $\vec{L}_f$, we find that

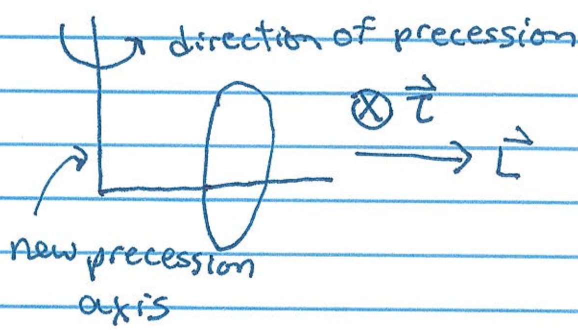
$$\vec{L}_f = L\hat{i} + \Delta t\,\tau\hat{j},$$

that is, the angular momentum picks up a y component! The picture of this is:

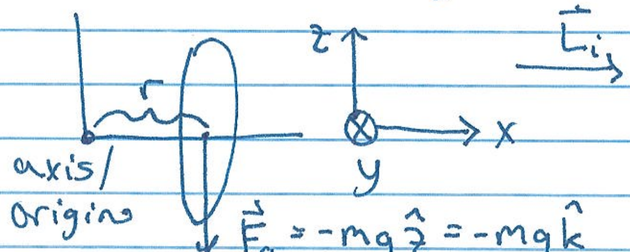


So, apparently, the angular momentum just rotates about the rope! We can continue this to a later time by asking what happens to $\vec{L}_f$ because of the torque, etc. A rotating wheel does not fall: if it did, the direction of the change of angular momentum would be inconsistent with Newton's second law!

This ~~exte~~ extremely weird rotational phenomena is called precession. If a torque acts parallel (or anti-parallel) to angular momentum, then it acts to accelerate the angular velocity. However, if a torque acts perpendicular to the direction of angular momentum, as in the case at hand, it acts to precess, or rotate, the angular momentum's direction about the axis of rotation that is perpendicular to both the torque and angular momentum. In this case, we have:



In the time that remains, I want to study the Newton's second law equation and see if we can massage it into an interesting form to solve for angular momentum $\vec{L}$, valid at all times. Let's start by putting down coordinates:



$\vec{F}_g = -mg\hat{z} = -mg\hat{k}$

With this setup, note that the vector $\vec{r}$ from the origin/axis to the center-of-mass of the wheel is:

$$\vec{r} = x\hat{i} + y\hat{j} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j},$$

where r is the distance from the axis to the center of the wheel and θ is the angle the vector makes with respect to the x-axis, in the x-y plane. Then, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F}_g = -mg\left(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}\right) \times \hat{k}$$

$$= -rmg\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right) = \left(-rmg\sin\theta\right)\hat{i} + \left(rmg\cos\theta\right)\hat{j}$$

Note that the direction of angular momentum is radial, away from the origin, so we can express it as:

$$\vec{L} = L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}$$

Note that because the torque acts perpendicular to angular momentum, the magnitude of angular momentum

is fixed, just like the magnitude of linear momentum is fixed if a force acts perpendicular to momentum.
Now, Newton's second law says that:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow (-rmg\sin\theta)\hat{i} + (rmg\cos\theta)\hat{j} = L\frac{d\cos\theta}{dt}\hat{i} + L\frac{d\sin\theta}{dt}\hat{j}$$

Let's focus on the $\hat{i}$ components:

$$-rmg\sin\theta = L\frac{d\cos\theta}{dt}.$$

Using the chain rule, we have

$$L\frac{d\cos\theta}{dt} = L\frac{d\cos\theta}{d\theta}\frac{d\theta}{dt} = -L\sin\theta\cdot\omega = -rmg\sin\theta$$

Canceling factors, we have: $\omega = \frac{rmg}{L}$, which is constant.

We can do the same analysis for the $\hat{j}$ component:

$$L\frac{d\sin\theta}{dt} = L\frac{d\sin\theta}{d\theta}\frac{d\theta}{dt} = L\cos\theta\cdot\omega = rmg\cos\theta$$

or again that $\omega = \frac{rmg}{L}$.

Therefore, the angular momentum under this precession is:

$$\vec{L} = L\cos\left(\frac{rmg}{L}t\right)\hat{i} + L\sin\left(\frac{rmg}{L}t\right)\hat{j}.$$

Note that this sweeps out a circle over time, of radius L at rate $\omega = \frac{rmg}{L}$. That is, precession is nothing more than uniform circular motion of angular momentum.

Have a good Thanksgiving!