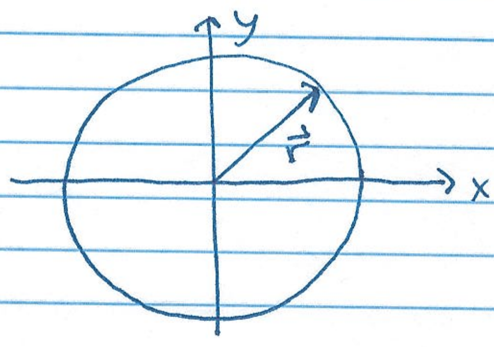
Lecture 32 Physics 101

Welcome back from Thanksgiving break! Hope you were able to relax for a few days. I haven't completed grading your exams, but they will be done in a few days. Also, remember that this week is the last week of conferences and labs, so make sure to attend! Finally, the homework that was assigned last Wednesday is due next lecture, December 4. After that, there are just a couple more homework assignments for the semester!

For the remaining lectures, we are going to study the general phenomena of oscillations. We've encountered oscillations at several points throughout this semester. First, when we introduced circular motion long ago, we had expressed the position vector $\vec{r}$ of an object undergoing circular motion as:

$$\vec{r}(t) = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$$

and this vector looks like:



and it revolves counterclockwise at a rate of ω rad/s. The radius of the circle is r.

Just before the break, last Wednesday, we introduced precession, or, the uniform circular motion of angular momentum due to a torque applied perpendicular to the direction of angular momentum. We had found that the resulting angular momentum vector $\vec{L}$ can be expressed as:

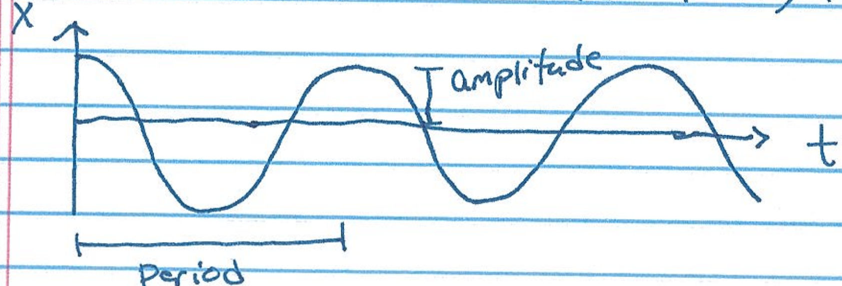
$$\vec{L}(t) = L\cos\left(\frac{\tau}{L}t\right)\hat{\imath} + L\sin\left(\frac{\tau}{L}t\right)\hat{\jmath},$$

where L is the magnitude of the angular momentum (constant in time) and τ is the magnitude of the torque applied perpendicular to angular momentum. This precession or "uniform circular motion" of angular momentum is <u>exactly</u> analogous to the uniform circular motion described earlier. If a force is applied exclusively perpendicular to velocity/momentum, then no work is done on the object (no speed is changed) and only the direction of motion is affected. So, precession and uniform circular motion are very much so the same phenomena, just manifest in different systems.

In what sense, however, are either of these systems "oscillating", or swung back and forth (in Latin)? Instead of considering the circle swept out as a function of x and y position as time passes, let's just focus on one coordinate; say, the x-coordinate of uniform circular motion as a function of time. That is, let's just focus on the function:

$$x(t) = r\cos\omega t.$$

What does this look like? Plotting it, we have:

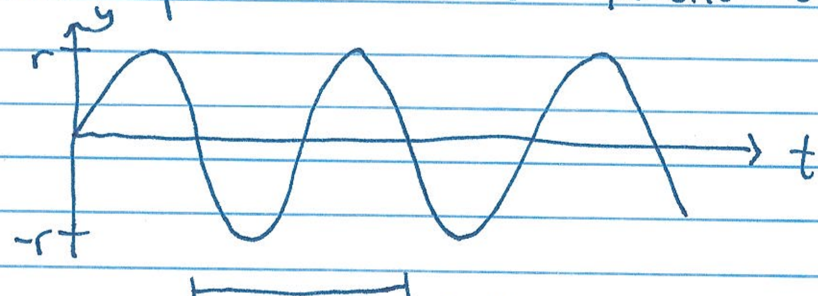


This position versus time graph illustrates clearly "oscillation" and the graph has a number of features that we give names to. First, the maximum distance of the oscillation away from the abscissa (horizontal axis) is called the amplitude of the oscillation. Because the maximum value of $\cos\omega t$ is 1, the amplitude is simply the coefficient of $\cos\omega t$; in this case, the distance r. Additionally, this oscillation repeats its pattern over a given length of time. The minimal length of time over which it repeats is called the period of oscillation. Colloquially, we often call this oscillation a wave, but waves (properly) exhibit more phenomena than a generic oscillation (and what you will discuss next semester). This oscillation that is controlled by cosine (or sine) is referred to as a sinusoidal oscillation.

Okay, we've got our bearings. Now, let's take a look at the y-component of uniform circular motion. We have the formula:

$$y(t) = r\sin\omega t,$$

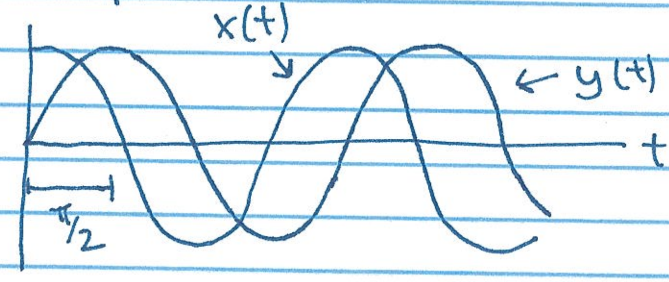
and a plot of this as a function of time is:



Period $T = \frac{2\pi}{\omega}$

Note that this oscillation has the same period and amplitude as for the x-coordinate. However, we say that $x(t)$ and $y(t)$ oscillations are $90°$ (or $\pi/2$) out-of-phase because their value at $t=0$ are different, by an angular factor of $\pi/2$.

To see this graphically, let's plot them on the same plot:



Over one period, ωt increases by 2π:

$$\omega(t+T) = \omega\left(t+\frac{2\pi}{\omega}\right) = \omega t + 2\pi$$

And so we also say that a period corresponds to an angular displacement of 2π (like going around a circle). Note that from $x(t)$ at $t=0$, one has to travel one quarter period in time to get to the same point of $y(t)$. One quarter period is $2\pi/4 = \pi/2$ radians, thus the $\pi/2$ out-of-phase.

Symbolically, let's manipulate $\sin\omega t$ into a form with $\cos\omega t$. To do this, note that for $y(t)$ to overlap $x(t)$, it needs to be moved <u>left</u> by a phase angle $\pi/2$. That is, at $t=0$, it needs to have have an argument that is smaller by $\pi/2$ than in $\sin\omega t$ form. This is just to say that

$$\sin \omega t = \cos\left(\omega t - \frac{\pi}{2}\right)$$

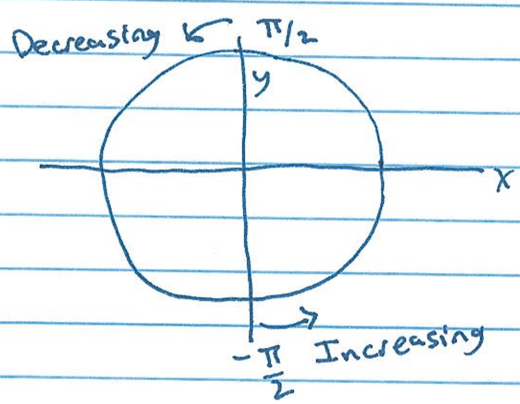
Let's see if this makes sense. First, at $t=0$, we have:

$$\sin 0 = \cos\left(0 - \frac{\pi}{2}\right) = 0 \text{, which is true.}$$

Now, regarding the "$-\frac{\pi}{2}$": this is more subtle. In principle we could have had the relationship

$\sin \omega t = \cos\left(\omega t + \frac{\pi}{2}\right)$, and this would also have

satisfied the requirement at $t=0$. However, it doesn't work immediately after. Let's go back to our unit circle:



At $t=0$, we have the two options: a phase of $\pi/2$ or $-\pi/2$, which I have illustrated. As time increases, we move around the unit circle in a counterclockwise manner.

This means that, from phase point $\pi/2$, for example, the x-component (cosine) decreases while from phase point $-\pi/2$ the x-component increases. Looking at the graph of $\sin \omega t$, as t increases, does $\sin \omega t$ increase or decrease? It increases, so therefore

$$\sin \omega t = \cos(\omega t - \pi/2).$$

Another symbolic way to identify this same result is simply by taking the derivative and then setting $t=0$.

Note that:

$$\frac{d}{dt}\sin\omega t\Big|_{t=0} = \omega\cos\omega t\Big|_{t=0} = \omega > 0$$

and $\frac{d}{dt}\cos(\omega t \pm \frac{\pi}{2})\Big|_{t=0} = -\omega\sin(\omega t \pm \frac{\pi}{2})\Big|_{t=0}$

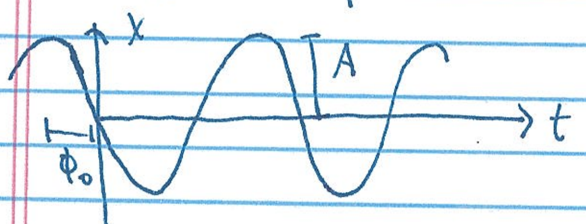
$= \mp\omega$, so for these to agree, again

we must take the "$-\pi/2$" phase factor.

Finally, I just want to note that a generic sinusoidal oscillation can be expressed as:

$$x(t) = A\cos(\omega t + \phi_0)$$

where A is the amplitude, ω is the angular frequency, and ϕ_0 is the phase:



Note that $+\phi_0$ means the wave is shifted left. This general form of the oscillation follows from the angle addition formula:

$$\cos(\omega t + \phi_0) = \cos(\phi_0)\cos(\omega t) - \sin(\phi_0)\sin(\omega t)$$

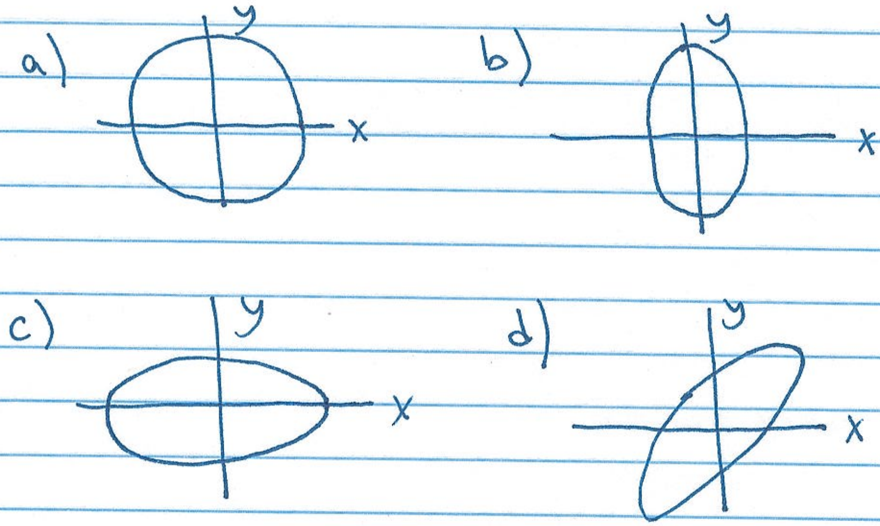
We'll use this formalism to understand physical phenomena on Wednesday. To end this lecture, I want to ask a couple of questions. First, we had said that Uniform circular motion is described by the vector:

$\vec{r}(t) = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}.$

Note that the amplitudes of the two components are identical, r. What if they are different, with $r_1 > r_2$:

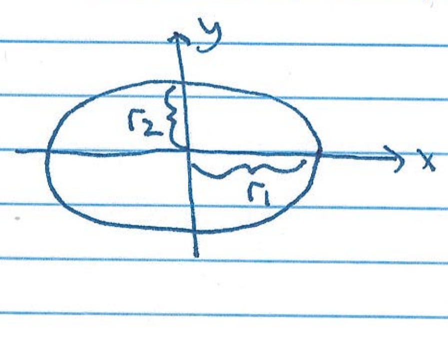
$$\vec{r}(t) = r_1\cos\omega t\,\hat{i} + r_2\sin\omega t\,\hat{j}$$

What does the trajectory of the object look like now?



I'll give you a minute to talk to your neighbors.

The answer is c. Because $r_1 > r_2$, the trajectory goes farther from the origin in the x direction than the y direction. Actually this trajectory is nothing more than an ellipse with semimajor axis r_1 and semiminor axis r_2:

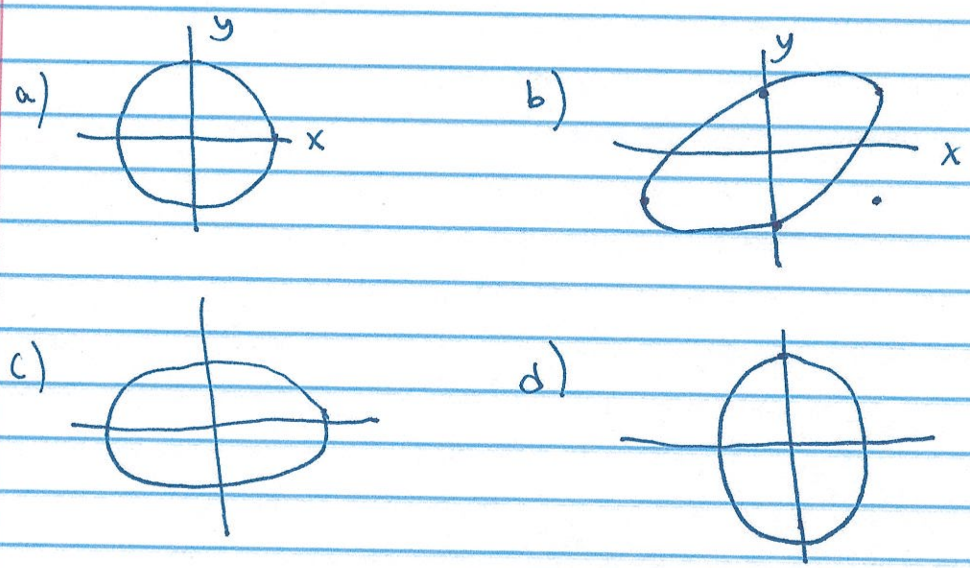


We had also said that $\sin\omega t$ is $-\pi/2$ out-of-phase of $\cos\omega t$ so the vector for uniform circular motion is:

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\cos(\omega t - \pi/2)\,\hat{\jmath}$$

This out-of-phase-ness is vital to produce a circular trajectory. However, what if the phase difference were only $-\pi/4$ instead of $-\pi/2$? That is, what trajectory would the vector

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\cos(\omega t - \pi/4)\,\hat{\jmath} \quad \text{sweep out?}$$



Talk with your neighbors for a second!

The answer is b. This is very tricky, but one way to see it is to evaluate $\vec{r}(t)$ at $\omega t = 0, \pi/2, \pi, 3\pi/2$ and connect the dots. Decreasing the phase difference between the x- and y-components rotates and smushes the trajectory. Consider what happens if that $-\pi/4$ is turned into 0. What is the trajectory now?

Have a good day! See you wednesday?