Lecture 33 Physics 101

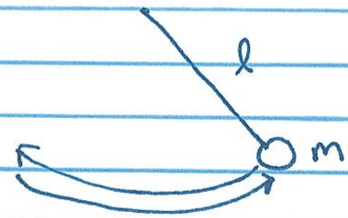
Welcome to this lovely Wednesday lecture! Please turn in homework and remember there is another homework assigned today due Friday.

Last lecture, we introduced the language of oscillations, specifically sinusoidal oscillations. We had said that an oscillating system can, in general, be expressed as:

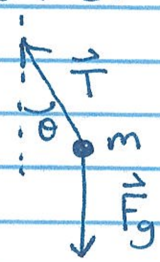
$$x(t) = A\cos(\omega t + \phi_0),$$

where A is the amplitude of oscillation, ω is the angular frequency of oscillation, and ϕ_0 is called the phase (factor) of the oscillation, which just quantifies where the object starts at $t=0$. Here, $x(t)$ denotes <u>any</u> thing that might oscillate: position, angular momentum, angle, etc., and a different physical system will have a different quantity that oscillates. In this lecture, we will use this general formula to analyze some familiar physical systems.

We'll start with the pendulum, which as I have illustrated before, is nicely exhibited in this room with a bowling ball and a rope. Let's remind ourselves of what this pendulum is doing. I release the pendulum from some initial position and then it swings back and forth, that is, oscillates like so:



On this figure, I have denoted the mass of the bowling ball as m and the length of the rope as l. To analyze the oscillation of the pendulum, let's draw the free-body diagram of the mass:



$\vec{T}$ is the tension in the rope and $\vec{F}_g = mg\hat{j}$ is the gravitational force.

As the mass oscillates, it remains a fixed distance l from the axis of oscillation. That is, the mass sweeps out the arc of a circle as it swings back and forth. As such, there is no motion perpendicular to this circle, so we know that forces that act in the direction of the axis of oscillation, i.e., centripetal forces, act only to change the direction of velocity of the mass.

So, the relevant force for oscillation is the force tangent to the circular arc, that is responsible for the ball speeding up and slowing down while it oscillates. What is this force? Well the tension is exclusively centripetal, so not that. The only relevant force then is a component of gravity that depends on the angle of the rope with respect to vertical:

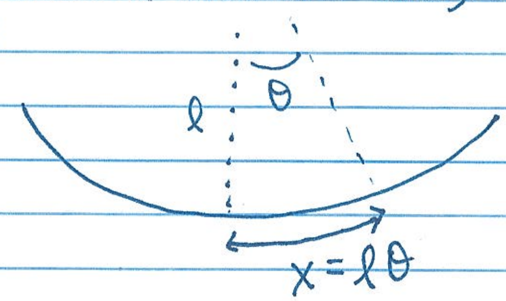
$F_{rel} = -mg\sin\theta$

Note the lack of vector: we can consider the oscillation as one-dimensional: just fixed to the arc of a circle. Also notice the "-" sign: this is a restoring force acting to always pull the ball toward $\theta = 0$ where the rope is vertical.

Now that we have this force, we want to use it to determine Newton's second law. Newton's second law is of course

$F = ma = m\dfrac{d^2x}{dt^2}$, but we need to figure out what the position "x" is.

Recall that the motion of the pendulum is along the arc of a circle, so the distance x is just some distance measured along the arc of this circle:



Now, let's take derivatives of $x = l\theta$:

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(l\theta) = l\frac{d^2\theta}{dt^2} \equiv l\ddot{\theta}$$

The length of the rope l is constant, so it just pulls out of the derivative. Then, our Newton's second law is:

$$-mg\sin\theta = ml\ddot{\theta}$$

This is a differential equation for the angle θ, the angular displacement from vertical, as a function of time. It is called a second-order differential equation because there are at most two derivatives of θ present in the equation. Further, it is a non-linear differential equation because a non-linear function of θ, namely $\sin\theta$, is present in the equation. As a non-linear second-order differential equation it is extremely challenging to solve (see the Wikipedia page on "Pendulum (mathematics)" for details).

So it might seem like we are up a creek without a paddle as they say, and can't go further. However, let's see if we can simplify this. If we only consider small angular displacements from vertical; that is, $\theta \ll 1$ in radians, then the $\sin\theta$ non-linear function can be simplified via Taylor expansion. For $\theta \ll 1$, $\sin\theta$ is approximately:

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} + \ldots \cong \theta,$$

which is a linear function of θ! With this assumption, our Newton's second law becomes:

$-mg\theta = ml\ddot{\theta}$, which is simple and can easily be solved.

As we are studying oscillations, well, simple harmonic oscillators, we will just make the ansatz that the solution to this differential equation can be written as:

$$\theta = A\cos(\omega t + \phi_0).$$

for some amplitude A, angular frequency ω, and phase ϕ_0. Let's just plug this into the differential equation and see what we find.

The second derivative of θ is:

$$\frac{d^2}{dt^2} A\cos(\omega t+\phi_0) = A\frac{d}{dt}\left(-\omega\sin(\omega t+\phi_0)\right)$$

$$= -A\omega^2\cos(\omega t+\phi_0),$$ so Newton's second law is:

$$-mg\left(A\cos(\omega t+\phi_0)\right) = -m A l\omega^2\cos(\omega t+\phi_0),$$

or, canceling factors becomes:

$$g = l\omega^2.$$

Note the miracle that happened here. We initially had a second order differential equation we had to solve. With our <u>sinusoidal</u> ansatz, we transmogrified the differential equation into a second-order <u>algebraic</u> equation that is trivial to solve:

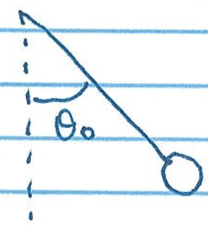
$$\omega = \sqrt{\frac{g}{l}}.$$

This general procedure of transforming a differential equation into an algebraic equation with a sinusoidal ansatz is called a <u>Fourier Transform</u>, after Joseph Fourier, a 19th century French mathematician. Fourier, among other things, first described the greenhouse effect.

So this Fourier transform immediately gives us the angular frequency, so our solution for the angle as a function of time is:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}\,t + \phi_0\right).$$

What about A and ϕ_0? The differential equation can't help us there, but the <u>initial conditions</u> of the pendulum can. Initially, if we hold the pendulum an angle θ_0 from vertical:



by energy conservation, we know that at any later time, the angle of the pendulum can <u>never</u> be larger than θ_0.

That is θ_0 is the <u>amplitude</u> of oscillation, and this amplitude is the angular displacement at $t=0$. Thus $\phi_0 = 0$ and

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{\ell}}\,t\right).$$

Whew! There are a few things to note:

- The velocity of the pendulum is

$$v(t) = \ell\frac{d\theta}{dt} = -\theta_0\sqrt{g\ell}\,\sin\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

Note that the velocity is $90°$ $(=\pi/2)$ out-of-phase with the position of the pendulum.

- We could have guessed $\omega \propto \sqrt{\frac{g}{\ell}}$ by dimensional analysis. In the statement of the problem, the only relevant dimensionful quantities were acceleration g, rope length ℓ, and mass m. The only way that these can combine into units of frequency (time^{-1}) is $\sqrt{g/\ell}$.

- There is a bit of numerology regarding the history of the pendulum and SI units. Note that the frequency f is

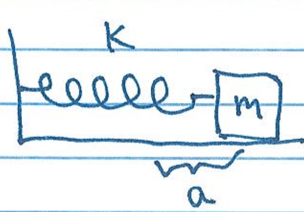
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$ and so the period of the

pendulum is
$$T = \frac{1}{f} = 2\pi\sqrt{\frac{\ell}{g}}$$

Why is $g = 9.8\,\text{m}/\text{s}^2$? Well, note that the period of the pendulum is independent of amplitude θ_0, so a pendulum of length ℓ is a very good, regular keeper of time. This has been known for at least 400 years. The length ℓ is very easy to measure, but g is more subtle. Wouldn't it be very convenient if the hard thing to measure, g, canceled the hard number to express, π? Indeed, note that

$\pi^2 = 9.8696\ldots$ eerily close to the SI value of g!

In the final minutes of this lecture, I want to briefly introduce the spring simple harmonic oscillator:



We have a mass m connected to a spring with constant k, initially displaced from equilibrium by a distance a. My first question to you is: using dimensional analysis, what is the angular frequency ω of the mass as it oscillates?

a) $\omega = \sqrt{ak}$, b) $\omega = \sqrt{\frac{k}{m}}$ c) $\omega = \sqrt{\frac{m}{k}}$ d) $\omega = a\sqrt{km}$

Let's solve Newton's second law and see what we find! The spring force, Hooke's law, is

$F = -kx$ and so Newton's second law is

$-kx = m\ddot{x}$. This is almost exactly similar to Newton's second law for the pendulum!

So, let's use our ansatz for simple harmonic oscillation to solve this:

$$x(t) = A\cos(\omega t + \phi_0) \;,\; \ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi_0)$$

$$\Rightarrow -kA\cos(\omega t + \phi_0) = -m\omega^2 A\cos(\omega t + \phi_0) \text{, or that}$$

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}}.$$

Further, if the spring is initially stretched by length a, conservation of energy states that it can never be larger than a. Therefore the amplitude is a and $\phi_0 = 0$ so

$$x(t) = a\cos\left(\sqrt{\frac{k}{m}}\,t\right).$$

That's it for today! See you friday for evaluations!