lecture notes useful? Donuts?

lec 34 1

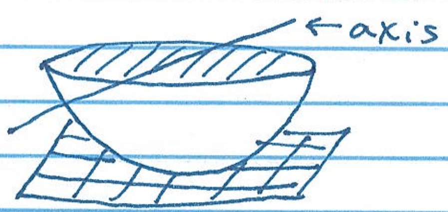
# Lecture 34 Physics 101

Welcome back to the penultimate meeting for this semester! Please turn in homework and the final homework is assigned, due Wednesday. I will have regular office hours through Wednesday, and then more irregular through the final. Please see Moodle for details on office hours and drop-in tutoring times. Also, this week has no conferences, and lab will be open for make-up, if necessary.

In this, our final contentful lecture of the semester, we are going to wrap up a couple loose ends, deriving some necessary results for homework and just introducing traveling waves, that you'll study more next semester. As we have for the past week, we are continuing to study oscillations, as modeled by the sinusoidal function:

$$x(t) = A\cos(\omega t + \phi_0),$$

where $A$ is the amplitude, $\omega$ is the angular frequency, and $\phi_0$ is the phase of oscillation. We've seen this formula applied in numerous cases already, and you'll see somemore in this final homework. Infact, this final homework requires you to find the period of oscillation of a solid hemisphere, when set upon its rounded end and slightly perturbed:



To be able to solve this problem, you will need to know both the center-of-mass of this hemisphere, as well as its moment-of-inertia about the axis that sits on the flat part (as illustrated).

Let's first find the center-of-mass. Let's call the total mass of the hemisphere $M$, and note that because it is, um, half of a sphere, its volume is

$$Vol = \frac{1}{2} Vol_{sphere} = \frac{2}{3}\pi R^3, \text{ where } R \text{ is its radius.}$$

Then, the density of the hemisphere is:

$$\rho = \frac{M}{Vol} = \frac{3}{2}\frac{M}{\pi R^3}.$$

Recall that the ~~location~~ position vector of the center of mass of an object is:

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm,$$ where $dm$ is a small mass and $\vec{r}$ is the position vector of that small mass.

Using the fact that we can write the small mass as:

$$dm = \rho\, dV,$$ where $dV$ is a small volume,

the center of mass is:

$$\vec{r}_{cm} = \frac{\rho}{M}\int \vec{r}\, dV = \frac{1}{Vol}\int \vec{r}\, dV.$$

Now, a few weeks ago, we identified the best/easiest way to analyze a sphere; that is by expressing the small volume in <u>spherical</u> <u>coordinates</u>. Then, we had found that

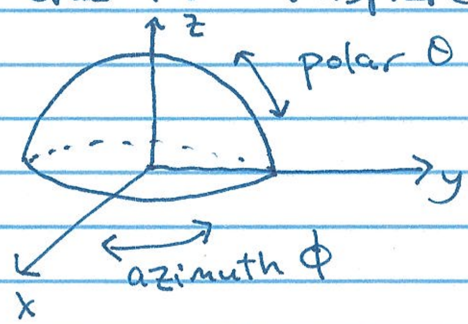
$$dV = r^2 \, dr \, d\cos\theta \, d\phi, \text{ where}$$

$r$ is the radial coordinate, $\theta$ is the polar angle, and $\phi$ is the azimuthal angle. So, for our hemisphere, we just need to calculate:

$$\vec{r}_{cm} = \frac{1}{Vol} \iiint \vec{r} \, r^2 \sin\theta \, dr \, d\theta \, d\phi .$$

This requires two things: figuring out what $\vec{r}$ is, and what the $r$, $\theta$, and $\phi$ bounds of integration are.

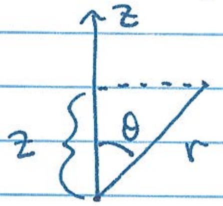
Let's draw the hemisphere more suggestively as:




Recall that the polar angle for a sphere (lines of latitude) range over $\theta \in [0, \pi]$, but for a hemisphere, we only get from the North Pole to the equator. So, for a hemisphere, $\theta \in [0, \pi/2]$. The azimuthal angle still ranges over $\phi \in [0, 2\pi]$ because for every $\theta$, we still can go all the way around a circle. Similarly, $r \in [0, R]$, the overall radius of the sphere. So, the center-of-mass calculation becomes

$$\vec{r}_{cm} = \frac{1}{Vol} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \vec{r} \, r^2 \sin\theta \, dr \, d\phi \, d\phi$$

What is the vector $\vec{r}$? Well, by the illustration earlier, we immediately see that the x- and y-components of the center-of-mass vector $\vec{r}_{cm}$ are 0. The only challenging component to evaluate is the z-component. Let's consider the illustration:




So we see that $z = r\cos\theta$. Plugging this into the expression for the center-of-mass, we see that

$$z_{cm} = \frac{1}{Vol}\int_0^R \int_0^{\pi/2} \int_0^{2\pi} (r\cos\theta)\sin\theta\, r^2\, dr\, d\theta\, d\phi$$

$$= \frac{1}{Vol}\left(\int_0^R r^3\, dr\right)\left(\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

We can then do these integrals one at a time. The $r$ and $\phi$ integrals are easy:

$$\left(\int_0^R r^3\, dr\right)\left(\int_0^{2\pi} d\phi\right) = \frac{R^4}{4}\cdot 2\pi = \pi\frac{R^4}{2}$$

and we can make a u-substitution for the $\theta$ integral. If we set

$u = \cos\theta$, then the $\theta$-integral becomes

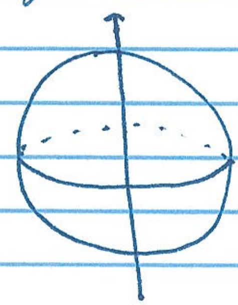
$$\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \int_0^1 u\, du = \frac{1}{2}, \text{ very simple!}$$

Putting it all together, we have:

$$z_{cm} = \frac{3}{2}\frac{1}{\pi R^3}\cdot \pi\frac{R^4}{2}\cdot\frac{1}{2} = \frac{3}{8}R.$$

That is, the z-coordinate of the center-of-mass is a distance of $\frac{3}{8}R$ from the flat side of the hemisphere. Does this make sense?

Now onto calculating the moment of inertia. This is actually very simple with an observation. Let's draw a sphere with an axis of rotation that passes through its center:



Let's say that this sphere has radius R and mass 2M. In this case, we know the moment of inertia is:

$$I_{sp} = \frac{2}{5}(2M)R^2.$$

Now for the tricky bit. Let's imagine that this sphere is composed of two hemispheres each of mass M, such that the axis of rotation lies on their flat faces. Then, it is clear that

$$I_{sphere} = 2I_{Hemisphere} = 2 \cdot \frac{2}{5}MR^2,$$ or that

the hemisphere moment of inertia is simply

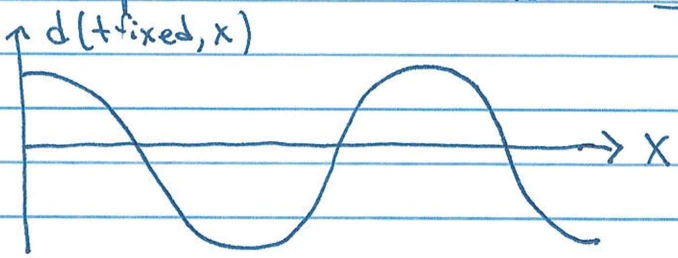
$$I_{hemi} = \frac{2}{5}MR^2!$$

Now for the last few minutes of this lecture, I want to briefly introduce traveling waves. We'll start this by a demonstration of waves on a string. There are a few things to note about the oscillations of the string. First and foremost, there is no net motion of the string, even though it appears that the string, or the wave on it is moving. What is happening is that each part of the string is oscillating up and down, but in a way that makes motion to the right appear, but actually doesn't happen.

So, if this string-wave is just oscillation of some flavor, we must be able to express the vertical displacement of any part of the string as:

$d(t) = A\cos(\omega t + \phi_0)$, where $A$ is the amplitude

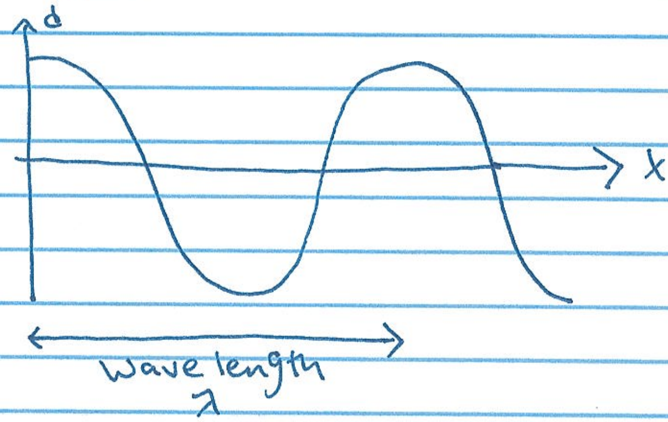
of displacement and $\omega$ is the angular frequency. This is simply related to how fast I move my hand up and down to create the wave. What is the phase, $\phi_0$? To answer this, let's consider a snapshot of the wave at a fixed time $t$. Then, the displacement as a function along the rope is:




Because we fix time to, say $t = t_0$, $\omega t$ is constant. Therefore the only way that the displacement can vary as a function of position is if the phase $\phi_0$ depends on position:

$$\phi_0 \equiv \phi_0(x).$$

What are the properties of this phase and its dependence on x? Note that the wave repeats itself after a minimal distance. We call the minimal distance over which the wave repeats itself the <u>wavelength</u>, $\lambda$:




Compare this to the definition of the period, $T$. Therefore, if the functional form of the displacement repeats every wavelength, the position-dependent phase must satisfy:

$$\phi_0(x+\lambda) = \phi_0(x) + 2\pi.$$

The simplest way to do this is if

$$\phi_0(x) = \frac{2\pi}{\lambda} x \equiv kx,$$ We call $k$ the "wave number",

and it has a form similar to that of the angular velocity,

$$\omega = \frac{2\pi}{T}.$$

Then, we have that the displacement of the rope as a function of time <u>and</u> position is

$$d(t,x) = A\cos(\omega t - kx).$$

The relative "−" sign is because the wave travels to the right. Compare this with our identification of $\phi_0$ for simple harmonic oscillation.

So, where does this perceived motion of the wave come from? Well, over time $T$, the rope's displacement repeats itself and in that time, the wave repeated itself over a distance $\lambda$. Then, the speed of the wave, that is, how fast the bumps appear to move right is:

$$V_{wave} = \frac{\lambda}{T} = \frac{\omega}{k} = V_{phase}$$

This particular wave speed is called the <u>phase velocity</u>. If we are able to express the angular frequency $\omega$ as a function of wave number $k$, then we can define another speed called the <u>group velocity</u>:

$$V_{group} = \frac{d\omega}{dk}$$

As a final mystery, recall that the kinetic energy of an object is:

$$E = \frac{1}{2} m v^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m}, \text{ related to momentum } p.$$

The speed of the object is:

$$v = \frac{d}{dp}\frac{p^2}{2m} = \frac{2p}{2m} = \frac{p}{m} = v.$$

Fascinating! Is energy related to $\omega$ and momentum to $k$?

That is it! See you Wednesday for review!