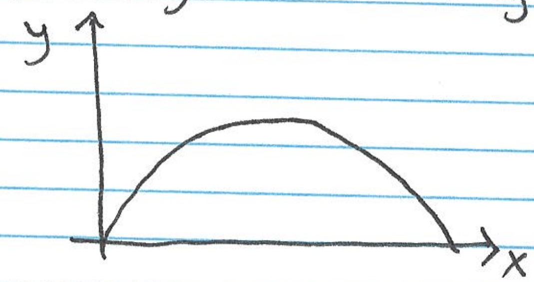
Lecture #6

Please turn in homework!

On Friday, we discussed and introduced the language of describing physics in multiple dimensions: vectors. This semester, we'll mostly restrict our analysis of physical systems to 2 dimensions; or, systems whose dominant properties are restricted to a plane. This will be sufficient to introduce nearly all of the interesting subtleties of physics in multiple dimensions and more features of physics in full three spatial dimensions will be introduced next semester.

We will first study the physics of projectile motion. A projectile is an object that flies freely, only influenced by the effects of gravity. More colloquially, a projectile can be a thrown baseball, a golf ball, the Vomit Comet airplane, or a whale that has unfortunately just materialized high in the atmosphere. We start studying projectiles because the physics it manifests is quite simple. As we discussed last week, a dimension is defined as an independent direction in space. This property of independence implies that for studying physics in multiple dimensions we can consider the properties of and physics of each dimension separately and then combine the analysis at the end. Projectile motion exhibits sufficiently interesting, yet simple phenomena in each dimension, which is why it's a good place to start.

First, let's motivate the physics of projectiles. We'll be studying physics applicable to our everyday, human-sized experience. On this scale, the surface of the Earth, like the Great Lawn, is, to very, very good approximation, flat. This direction along the ground will be one of our dimensions. The other dimension we consider is vertical: projectiles (like a baseball) travel both along the ground direction and up (and down) through the air. So, the two dimensions that we study can nicely be drawn on a page or blackboard. For example, the trajectory of a ball that we throw across the softball field on the great lawn might look like:



Here, x is the distance along the ground and y is the height of the ball and you throw it to the right from the origin.

Let's identify the physics in each of these dimensions. First, the vertical dimension. Again, I want to emphasize that because vertical and horizontal are independent dimensions, we can analyze them separately. From our everyday experience, gravity acts exclusively in the vertical direction. To good approximation (so far), gravity enacts a constant acceleration of g ($\cong 10\,m/s^2$) toward the ground. We had already introduced the formula for position as a function of time for an object undergoing constant acceleration. That is, for a negative, constant acceleration of magnitude g, the height y as a function of time is:

$$y(t) = -\frac{g}{2}t^2 + v_{0y}t + y_0,$$

where v_0 and y_0 are initial velocity and height, respectively. Here, "initial" means time $t=0$.

Note that I am careful to denote the velocity as the initial y-component of velocity, v_{oy}, as that is all that is relevant for the formula for the height.

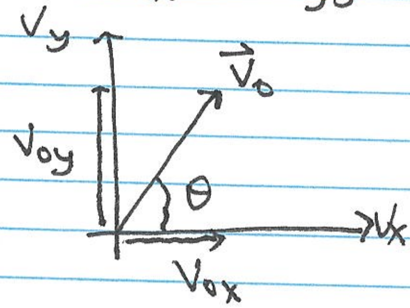
For horizontal motion, we want a corresponding formula for $x(t)$. Is there any acceleration horizontally? That is, if I drop a ball, does it spontaneously accelerate along the direction of the ground? Nope, or at least I don't think so! So, if there is no acceleration $(a=0)$ in the x-direction the expression for the horizontal position as a function of time is simple:

$x(t) = v_{ox}t + x_o$, where v_{ox} is the initial (time $t=0$) velocity in the x-direction. So, we have the two equations for horizontal and vertical position:

$$x(t) = v_{ox}t + x_o \quad , \quad y(t) = -\frac{g}{2}t^2 + v_{oy}t + y_o$$

For the initial velocities v_{ox} and v_{oy}, note that these are simply the x- and y-components of the two-dimensional initial velocity vector, $\vec{v}_o$. This can be expressed and illustrated as:

$$\vec{v}_o = v_{ox}\hat{i} + v_{oy}\hat{j} = (v_{ox}, v_{oy})$$



This vector can equivalently be expressed in terms of the magnitude of $\vec{v}_o$, v_o, and the angle θ above the horizontal as:

$$v_{ox} = v_o \cos\theta, \quad v_{oy} = v_o \sin\theta.$$

That is, our system of equations reads:

$$x(t) = v_0 \cos\theta\, t + x_0, \quad y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta\, t + y_0.$$

By taking derivatives, we can find velocities and accelerations in these two dimensions.

Again, I want to emphasize a consequence of the independence of the horizontal and vertical dimensions. Every object, regardless of its velocity vector initially, accelerates with g by gravity (ignoring air resistance). This result prompted a famous example of astronauts on the moon (where there is no atmosphere) to demonstrate that a feather and a hammer would hit the Moon's surface at the same time when dropped from the same height.

This also segues into an example we'll study for much of the rest of lecture. Here's the setup. A monkey is hanging from a branch a distance l from you horizontally and a height h above you. This monkey has a very loudly growling stomach so you think you can give it one of your bananas. This is a very smart (and hungry!) monkey so it doesn't like things thrown at it. In defense, the monkey releases its grip on the branch at the exact moment that anyone throws something at it. To ensure that the ~~monkey~~ banana hits the monkey, where should you aim when you throw it?

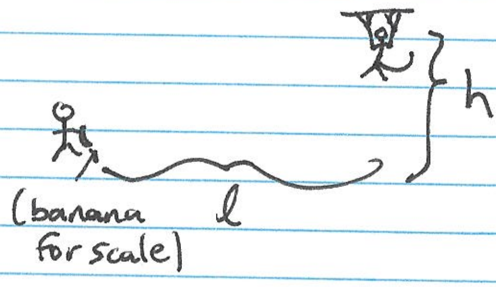
a) above the original location of the monkey

b) at the original location

c) below the original location

Talk to your neighbors about this! I'll give you a minute.

To answer this problem, let's first draw a picture:



Let's first write down the kinematic equations for the banana. With you located at the spatial origin and throwing the banana at time $t=0$, the trajectory of the banana is

$$y_b(t) = -\frac{g}{2}t^2 + v_0 \sin\theta\, t \,, \quad x_b(t) = v_0 \cos\theta\, t .$$

We assume that you throw the banana with speed v_0 at an angle θ above horizontal. Now, the monkey just drops itself from the branch at time $t=0$, so its horizontal position is constant in time: $x_m(t) = \ell$. Its height, on the other hand, is:

$$y_m(t) = -\frac{g}{2}t^2 + h \,,$$ note that the monkey's

initial velocity is 0: it drops from rest.

Now, if the banana hits the monkey at some time T, this means that the horizontal and vertical components of the positions of the monkey and banana are identical. For the horizontal positions, this enforces:

$$x_b(T) = x_m(T) = v_0 \cos\theta\, T = \ell$$

or $T = \dfrac{\ell}{v_0 \cos\theta}$. Now, plugging this into the

equations for the heights of the monkey and banana and setting them equal, we have:

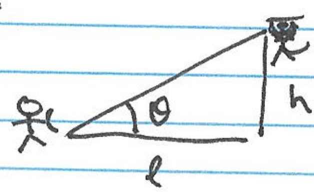
$$y_b(T) = y_m(T) \Rightarrow -\frac{g}{2}\left(\frac{\ell}{v_0\cos\theta}\right)^2 + v_0\sin\theta\,\frac{\ell}{v_0\cos\theta}$$

$$= -\frac{g}{2}\left(\frac{\ell}{v_0\cos\theta}\right)^2 + h$$

Massaging this expression we have that the terms proportional to g cancel each other! The way we interpret this is that the banana and the monkey "fall" for the same distance. Canceling these terms, we find that the banana hits the monkey if:

$$\frac{v_0 \ell \sin\theta}{v_0\cos\theta} = h \quad \text{or} \quad \tan\theta = \frac{h}{\ell}.$$

Remember, θ is the angle above the horizontal of the initial velocity, $\tan\theta$ is the ratio of the height to the length of a right triangle, and h/ℓ is exactly the ratio of the height and length of the triangle formed from you, the monkey, and the ground:



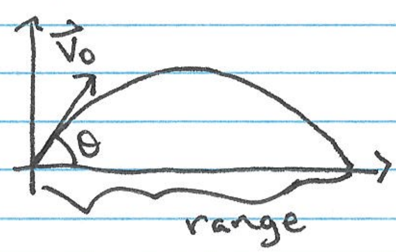
That is, you should aim right at the monkey!

Just our luck, we can test this out here today! Before we do, I want to tell you a couple quick stories. First, there's the adage that if it wiggles, it's biology. If it stinks, it's chemistry. If it's dirty, it's geology. And if it doesn't work, it's physics.

Also, as I've mentioned before, I'm a theoretical particle physicist, which means I stay far away from experiment. There's a phenomena called the "Pauli Effect" named after the theorist Wolfgang Pauli who is perhaps most famous for the Pauli exclusion principle. Pauli was notorious for destroying any and all experimental appartuses he came in contact with. A statement of the Pauli effect is that Pauli and a working experiment can not be found in the same room. Apparently this effect was extremely strong. Once an experiment in Goettingen, Germany, failed for some unknown reason, completely randomly. The experimentalists were baffled, until they learned much later that at almost the same time of the failure, Pauli was transferring trains... in Goettingen!

So hopefully I have better luck than Pauli!

Finally, I want to present an estimate and plausibility argument for the range formula. That is, given level ground and an initial velocity v_0 an angle θ above the horizontal, how far does a projectile travel? The picture is:



The range is a distance, so we can use dimensional analysis to determine its dependence on the given quantities.

The only dimensionful quantities in the problem are the initial speed v_0 and acceleration g. The range r is formed by some product of these quantities raised to powers a and b:

$$r = v_0^a g^b.$$

We can determine a and b by matching units.

The units of r are meters m and the units of $v_0^a g^b$ are:

$$[v_0^a g^b] = \frac{m^a}{s^a}\frac{m^b}{s^{2b}} = m^{a+b} s^{-a-2b}.$$

If this is to reduce to just meters, we know that the second units must be eliminated; or that

$-a-2b=0$. Further, a single distance unit means

that: $a+b=1$.

These two equations can be solved by first adding them together:

$$(-a-2b)+(a+b) = 0+1 \Rightarrow -b=1 \Rightarrow b=-1$$

and then it follows that $a=2$. That is, the range is proportional to:

$$r \propto \frac{v_0^2}{g}.$$

What about dependence on the angle θ? Well, we would have to solve and reorganize the kinematic equations, but we can note two limits. First, if $\theta=0$, the projectile is shot parallel to the ground from the ground, so has 0 range. Also, if $\theta=90°$, the projectile travels straight up vertically and lands where it started. Again, this has 0 range. These considerations suggest that r is proportional to

$$r \propto \frac{v_0^2 \sin(2\theta)}{g}.$$ When $\theta=0°$, $\sin 0°=0$ and when $\theta=90°$, $\sin(2\cdot 90°)=\sin 180°=0$.

We need a bit more work to justify it, which you can find in the book, but the range formula is actually just this:

$$\boxed{r = \frac{v_0^2 \sin(2\theta)}{g}}$$