Phys 101 Lecture 7

Please turn in homework!

One of my favorite pasttimes as a kid on long car trips was staring at the other cars on the road and watching them pass by. It's a mesmerizing thing: your car could be traveling 70 miles per hour with respect to the ground, but the car next to you could appear to be at rest. Or, if your parent(s) had a particularly heavy lead foot the cars next to you could appear to be traveling backward, even though everyone was moving forward. Looking across the road, to the oncoming traffic, they would zip by at huge speeds, much faster, it would appear, than we were traveling individually. How do we make sense of these relative velocities and speeds? We can exploit vectors and their formalism to study this problem.

Let's analyze this problem, starting in one dimension. I'm imagining that I'm in the back seat of my parent's car, on the long drive to my grandparents. While driving through the deserts of the western US, there's a lot of time to think about physics, so I'm wondering how fast the cars in the other lanes appear out my window. Let's say that the velocity of my car is $\vec{v}_{me}$, while the velocity of the truck with Florida plates is, $\vec{v}_{truck}$. Hold on a second; we have a bit more to define. Just as we needed to specify an origin from which to measure distances when I walked across the well, we need to define what we measure velocities with respect to.

Velocity is defined as a difference of displacement over change in time:

$$\vec{v}(t) = \frac{d}{dt}\vec{d}(t)$$

so velocity is independent of the spatial origin we choose. A different spatial origin just corresponds to some displacement of our position by some constant vector $\vec{d}_0$:

$$\vec{d}(t) \to \vec{d}(t) + \vec{d}_0$$

Because $\vec{d}_0$ is independent of time, velocity is unaffected:

$$\vec{v}(t) \to \frac{d}{dt}\left(\vec{d}(t) + \vec{d}_0\right) = \frac{d}{dt}\vec{d}(t) = \vec{v}(t).$$

However, last week we had done the gedankenexperiment in which we considered sitting in a very smooth train traveling at a constant velocity. Recall that we couldn't tell if the train was actually moving, if our eyes were closed or we weren't looking out the window.

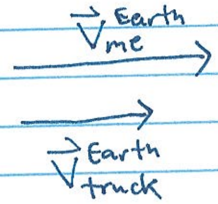
An interpretation of this is that there is nothing special about a particular absolute constant velocity. All that matters is relative velocities. So, whenever we talk about a velocity we need to specify with respect to what. Those objects that have 0 ~~net~~ velocity with respect to one another are said to define a frame of reference, or just frame.

So, when we say that we have velocity of $\vec{V}_{me}$, we need to define the frame in which this velocity is defined. Typically, it's the velocity with respect to the Earth, so for analyzing the problem at hand, we denote our velocity and the velocity of the truck from the frame of the Earth as:

$$\vec{V}_{me}^{Earth}, \quad \vec{V}_{truck}^{Earth}$$

Using these quantities, can we determine the velocity of the truck I would see out my window? That is, what is the velocity of the truck in my reference frame, $\vec{V}_{truck}^{me}$?

Let's draw some vectors to express this setup:

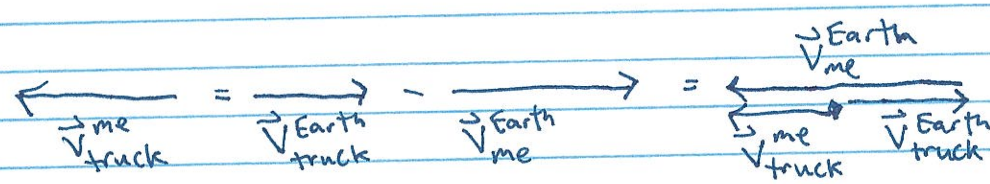


From these, we want to determine $\vec{V}^{me}_{truck}$. Let's think about how we get into my reference frame.

Imagine that you're just standing on a curb and a car drives by to your right (another Gedanken-experiment!). That is, in <u>your</u> frame the car drives right. ~~In the car's frame, however, the~~ Another way to imagine that the car moves to the right is not that you are at rest, but that the car is at rest and you are moving to the left! That is, to move from one frame to another, you need to <u>subtract</u> the relative velocity of the two frames.

That is, $\vec{V}^{me}_{me} = 0$, as I am at rest by definition in my frame. To determine the velocity of the truck in my frame, I just subtract my velocity with respect to Earth from the truck's velocity to the Earth:

$\vec{V}^{me}_{truck} = \vec{V}^{Earth}_{truck} - \vec{V}^{Earth}_{me}$, or, in arrow notation:



I can always determine another relative velocity from two velocities in a single frame:

$$\boxed{\vec{V}^{frame2}_{obj} = \vec{V}^{frame1}_{obj} - \vec{V}^{frame1}_{frame2}}$$

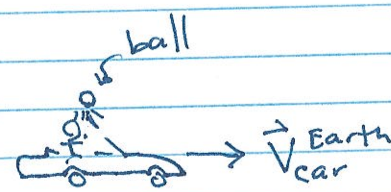
This is true as a vector equation, so is the rule for relating two frames in any number of dimensions.

Using this formalism let's analyze the following system. Imagine that you are sitting in the backseat of a convertible, with the top down, (wayfarers on) traveling at a constant velocity with respect to the Earth. You throw a ball vertically in the car's frame. What happens? Does the ball:

a) land behind you b) land in front of you

c) land right on you

A picture of this is:



What do you think? Talk to your neighbors!

let's analyze this systematically. We are given the following: $\vec{V}_{car}^{Earth} = V_{car}\hat{i}$, where V_{car} is the car's speed and we assume it is moving along the x-axis. Also, the ball's velocity in the frame of the car is: $\vec{V}_{ball}^{car} = (-gt + V_{ball})\hat{j}$.

Recall, in the car's frame, the ball travels vertically, in the y-direction. The ball's initial velocity in this frame is V_{ball}, and it, of course, undergoes acceleration $-g$ due to gravity.

Now, what is the velocity of the ball with respect to the Earth? From our master relative velocity formula, we have:

$$\vec{V}_{ball}^{Earth} = \vec{V}_{ball}^{car} - \vec{V}_{Earth}^{car}$$

This is a bit weird, as we know $\vec{v}_{car}^{\,Earth}$, but need $\vec{v}_{Earth}^{\,car}$. No fear; these are simply opposites of each other!

$$\vec{v}_{car}^{\,Earth} = -\vec{v}_{Earth}^{\,car} \quad \text{(Can you convince yourself of this?)}$$

Therefore, the velocity of the ball with respect to the Earth is:

$$\vec{v}_{ball}^{\,Earth} = \vec{v}_{ball}^{\,car} - \vec{v}_{Earth}^{\,car} = \vec{v}_{ball}^{\,car} + \vec{v}_{car}^{\,Earth} = v_{car}\hat{\imath} + (-gt + v_{ball})\hat{\jmath}.$$

Now, we ~~ch~~ want to determine the position of the ball after it has traveled up and come down to the vertical location of the car. To determine the displacement of the ball with respect to the earth, we simply integrate its velocity:

$$\vec{d}_{ball}^{\,Earth}(t) = (v_{car}t)\hat{\imath} + \left(-\frac{g}{2}t^2 + v_{ball}t\right)\hat{\jmath}$$

I've set the initial position at $t=0$ to be the 0 vector. The ball starts going up at $t=0$ and comes back to vertical displacement of 0 when:

$$-\frac{g}{2}t^{\cancel{2}} + v_{ball}\cancel{t} = 0 \Rightarrow t = \frac{2v_{ball}}{g} \equiv T_{y=0}$$

What is the ball's x-position at this time? We simply plug in $t = T_{y=0}$ to find:

$$x_{ball}(T_{y=0}) = v_{car}T_{y=0} = \frac{2v_{ball}v_{car}}{g}.$$

What is the car's x-position at this time? It's identical because, by the vector nature of velocity, the car and the ball have the <u>same</u> x-component of velocity! That is,

$$x_{car}(T_{y=0}) = \frac{2v_{ball}v_{car}}{g}.$$

That is, because both the x- and y- component of position of the ball and car are identical @ $T_y = 0$, the ball lands in my lap!

Let's test this out in a demo!

Also, what shape would you see as the trajectory of the ball (you being at rest with the Earth)? Note the ball's x-component of position is:

$$x_{ball}(t) = v_{car} t \quad \text{or that} \quad t = \frac{x_{ball}}{v_{car}}$$

Plugging this time into the expression for the y-component of the ball's position, we find

$$y_{ball}(t) = -\frac{g}{2} t^2 + v_{ball} t = -\frac{g}{2} \frac{x_{ball}^2}{v_{car}^2} + \frac{v_{ball} x_{ball}}{v_{car}}$$

That is, the height of the ball that you see as a function of its horizontal position is just a parabola:

$$y_{ball}(x_{ball}) = -\frac{g}{2} \frac{x_{ball}^2}{v_{car}^2} + \frac{v_{ball} x_{ball}}{v_{car}}$$

The independence of perpendicular spatial directions is extremely profound and produces concrete predictions we can test with experiment!