Phys 101 Lecture 8

Please turn in homework!

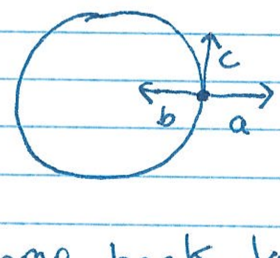
In this lecture, we are going to describe the kinematics of a very common situation that one encounters in physics. We have discussed the kinematics and description of linear motion, generalized that to multiple dimensions with vectors, and now we will discuss circular motion. Precisely what we will study in this lecture is the kinematics of traveling in a circle, but this analysis is applicable broadly to any motion that is non-linear: merry-go-rounds, a right turn in a car, a loop-the-loop roller coaster, etc.

I want to start this lecture with a question that we'll address throughout lecture. Imagine that you are on a merry-go-round, giggling with your friends. The merry-go-round is spinning/rotating at a constant speed. ~~Arg~~ Are you accelerating? And if so, in what direction are you accelerating?

a) No acceleration   b) Yes, accelerating outward

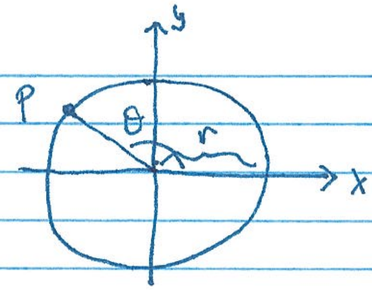
c) Yes, accelerating inward   d) Yes, accelerating forward

For b, c, ~~a~~ and d, the direction of acceleration of you on the merry-go-round is:



Discuss with your neighbors for a couple minutes!

We'll come back later to answer this definitively, but now we'll set up the language to describe circular motion. Let's first identify positions on a circle, with a convenient origin:



We have drawn a circle of radius r with the origin at the center of the circle. A point/position $\vec{p}$ on the circle can be represented in Cartesian coordinates (x, y space) as:

$\vec{p} = (r\cos\theta, r\sin\theta)$, where θ is the angle as measured above the +x axis, as illustrated. At this point, there is no motion; this is just a point, static, unchanging. If we move in a circle, our distance from the origin remains unchanged, but the angle θ changes in time. That is, we consider time dependence of

$$\vec{p}(t) = \left(r\cos\theta(t), r\sin\theta(t)\right), \text{ with } r \text{ a } \underline{\text{constant}} \text{ radius.}$$

What is the simplest temporal dependence for the angle $\theta(t)$? Just as we studied for motion in one dimension, the simplest motion is linear in time:

$$\theta(t) = \omega t + \theta_0$$

We'll mostly restrict to this case in this lecture. This angular time dependence is called constant angular velocity motion: ω is called the angular velocity as it is the rate that $\theta(t)$ changes:

$$\frac{d}{dt}\theta(t) = \omega.$$ θ_0 is a constant, the original location in angle of your position.

So, we'll study position that changes in time as:

$$\vec{p}(t) = \left(r\cos(\omega t+\theta_0), r\sin(\omega t+\theta_0)\right) = r\cos(\omega t+\theta_0)\,\hat{i} + r\sin(\omega t+\theta_0)\,\hat{j}.$$

Given this position, let's do our standard analysis of finding velocity and acceleration. Velocity is just the time derivative of position:

$$\vec{v}(t) = \frac{d}{dt}\vec{p}(t) = \frac{d}{dt}\Big(r\cos(\omega t+\theta_0)\Big)\hat{\imath} + \frac{d}{dt}\Big(r\sin(\omega t+\theta_0)\Big)\hat{\jmath}.$$

We need to take derivatives of cosine and sine to find velocities. To do this, we exploit the chain rule of derivatives. To take the derivative of a function of a function, the rule is:

$$\frac{d}{dt} f\big(g(t)\big) = \frac{df}{dg}\frac{dg}{dt}.$$

For the x-component of velocity, we want to take the derivative:

$\frac{d}{dt}\cos(\omega t+\theta_0)$. Let's call $f(\theta) = \cos\theta$

and $\theta(t) = \omega t+\theta_0$. Then, the derivative is:

$$\frac{d}{dt}\cos(\omega t+\theta_0) = \frac{d\cos\theta}{d\theta}\frac{d}{dt}(\omega t+\theta_0) = \omega\frac{d\cos\theta}{d\theta}$$

So, what's the derivative of $\cos\theta$? The answer is, which I won't explain in more detail here,

$\frac{d}{d\theta}\cos\theta = -\sin\theta$. Therefore, we have

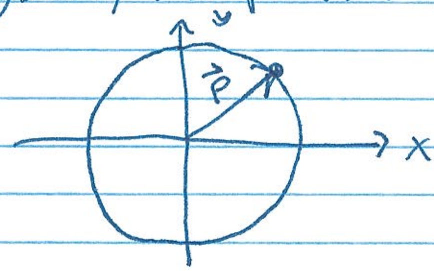
$$\frac{d}{dt}\cos(\omega t+\theta_0) = -\omega\sin(\omega t+\theta_0).$$

Similarly, $\frac{d}{dt}\sin(\omega t+\theta_0) = \omega\cos(\omega t+\theta_0)$.

It then follows that the expression for velocity when moving at constant speed around a circle is:

$$\vec{v}(t) = \big(-\omega r\sin(\omega t+\theta_0)\big)\hat{\imath} + \big(\omega r\cos(\omega t+\theta_0)\big)\hat{\jmath}.$$

It's interesting to pause here for a second and think for a bit about what this velocity is telling us. First, recall that, in the prescribed coordinate system, the position vector lies along a radial line:



At this point on the circle, what is the velocity vector?

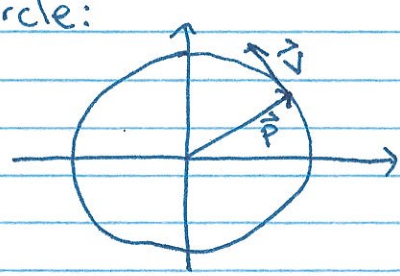
To answer this question, Let's use a trick from freshman geometry. Let's schematically denote the position vector as:

$$\vec{p}(t) = a\hat{i} + b\hat{j} = (a, b).$$

a and b are the cosine and sine bits, but for this argument that is just distracting. In terms of a and b, now, the velocity vector is:

$$\vec{v}(t) = \omega(-b\hat{i} + a\hat{j}) = \omega(-b, a) \propto (-b, a).$$

If I have two vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (-b, a)$, what angle do they make with one another? 90°! Therefore, if we are moving counterclockwise around the circle, the velocity vector is <u>tangent</u> to the circle:



This orthogonality can also be captured in a <u>dot product</u>. The dot product of two vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (c, d)$ is defined to be:

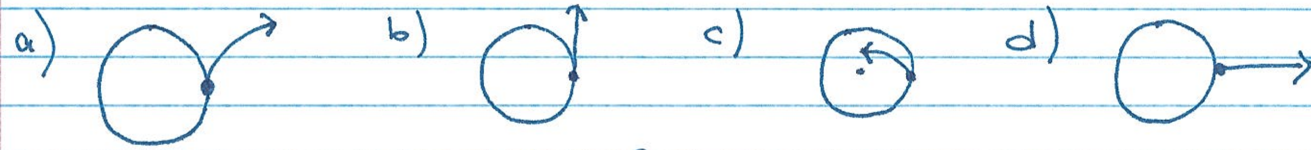
$\vec{v}_1 \cdot \vec{v}_2 = ac + bd$. If the dot product is 0, the vectors are orthogonal. Let's see this for position and velocity:

$$\vec{p}(t) \cdot \vec{v}(t) = (r\cos(\omega t + \theta_0), r\sin(\omega t + \theta_0)) \cdot (-\omega r\sin(\omega t + \theta_0), \omega r\cos(\omega t + \theta_0))$$

$$= -\omega r^2 \cos(\omega t+\theta_0)\sin(\omega t+\theta_0) + \omega r^2 \sin(\omega t+\theta_0)\cos(\omega t+\theta_0)$$

$$= 0!$$

Now, I have a question for you. Say an object is traveling at constant speed in a circle, for example, by being swung with a string. If the string is cut, how does the ball/object travel afterward?



Talk amongst your neighbors for a second!

We can see what happens in a demo/video! Also, consider a slingshot. You twirl the sling round and round and then let go. To hit a target in front of you, how and when should you release it?

Finally, let's get back to where we started, and address acceleration. If an object is moving in a circle at constant speed, is it accelerating? That is, is its velocity changing in time? We said that the speed was constant, so by definition the magnitude of velocity is not changing. However, that's not the only way to change velocity! As a vector, changing velocity can mean changing its magnitude and/or direction! Clearly the direction of velocity is changing as you move in a circle (otherwise you wouldn't, well, move in a circle). So, we're definitely accelerating.

We can just take another derivative of velocity to determine acceleration, but it's useful to take a step back first, and think about the direction of acceleration. To do this, let's return to our gendankenexperiment space. First, imagine that you are sitting in a car that is at rest. Now, the driver accelerates forward, by pressing on the gas pedal. What do you feel? What is your response to a forward acceleration? You are pushed backward; that is, you feel a push in the opposite direction to acceleration. An analogous thing happens if the driver now brakes. Acceleration is backward (forward motion slowing down), yet you are push forward. Keep this in mind.

Now, imagine that you are sitting at the edge of a merry-go-round that is being pushed at a constant rate by delinquint fourth graders. Just thinking about what you would feel, how and in what direction do you feel a push? So, from our thought experiment in the car, what is the direction of acceleration?

Let's now calculate acceleration from the derivative of velocity, we have:

$$\frac{d}{dt}\vec{v}(t) = \vec{a}(t) = \frac{d}{dt}\left(-\omega r\sin(\omega t+\theta_0),\ \omega r\cos(\omega t+\theta_0)\right)$$

$$= \left(-\omega^2 r\cos(\omega t+\theta_0),\ -\omega^2 r\sin(\omega t+\theta_0)\right)$$

But wait! Recall that the position vector for moving around the circle was:

$$\vec{p}(t) = \left(r\cos(\omega t+\theta_0),\ r\sin(\omega t+\theta_0)\right).$$

Therefore, acceleration is just: $\vec{a}(t) = -\omega^2\vec{p}(t)$

As a real number, $\omega^2 > 0$ and so acceleration points in the opposite direction as the position vector. As $\vec{p}(t)$ pointed from the center of the circle to its perimeter, acceleration points from the perimeter to the center! Thus, this acceleration is "center-seeking", or centripetal acceleration.

We'll end the lecture with one more observation.

Let's evaluate the magnitudes of velocity and acceleration and see if there's a relationship between them. Recall that the velocity vector was:

$$\vec{v}(t) = (-\omega r\sin(\omega t+\theta_0), \omega r\cos(\omega t+\theta_0))$$

Its magnitude, via Pythagoras, is:

$$|\vec{v}| = \sqrt{(\omega r\sin(\omega t+\theta_0))^2+(\omega r\cos(\omega t+\theta_0))^2} = \omega r,$$

because $\cos^2+\sin^2=1$. As such, ω is called the "angular velocity." What about acceleration? we have

$$|\vec{a}| = \sqrt{(\omega^2 r\cos(\omega t+\theta_0))^2+(\omega^2 r\sin(\omega t+\theta_0))^2} = \omega^2 r.$$

Note that the angular velocity is just: $\omega = \frac{|\vec{v}|}{r}$. Plugging this into the expression for acceleration, we have:

$$|\vec{a}| = \omega^2 r = \left(\frac{|\vec{v}|}{r}\right)^2 r = \frac{|\vec{v}|^2}{r}.$$

This will be a very useful result. The magnitude of centripetal acceleration is velocity squared divided by the radius of the circular motion.

That's it; have a good weekend!