Problem Set 1

Phys 342

Due February 7

Exercises, Due Friday, February 7

1. In lecture, we introduced the derivative matrix $\mathbb{D}$ defined as the derivative acting on a grid with spacing Δx. This matrix has the form:

$$\mathbb{D} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & \frac{1}{2\Delta x} & 0 & \cdots \\ \cdots & -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} & \cdots \\ \cdots & 0 & -\frac{1}{2\Delta x} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

with only non-zero entries immediately above and below the diagonal. In this problem, we will just study the 2×2 and 3×3 discrete derivative matrices.

(a) Explicitly construct 2×2 and 3×3 discrete derivative matrices, according to the convention above.

(b) Now, calculate the eigenvalues of both of these matrices. Note that there should be two eigenvalues for the 2×2 matrix and three eigenvalues for the 3×3 matrix. Are any of the eigenvalues 0? For those that are non-zero, are the eigenvalues real, imaginary, or general complex numbers?

(c) Determine the eigenvectors of this discrete derivative matrix, for each eigenvalue. Don't worry so much about the normalization of the eigenvectors; for simplicity, you can just assume that the first element of all the eigenvectors is 1.

(d) Now, consider exponentiating the discrete derivative matrix to move a distance Δx. Call the resulting matrix $\mathbb{M}$:

$$\mathbb{M} = e^{\Delta x\, \mathbb{D}}\,. \tag{2}$$

What is the result of acting this exponentiated matrix on each of the eigenvectors that you found in part (c)?
Hint: Consider the Taylor expansion of this exponential. How does $\mathbb{D}^n$ act on an eigenvector of $\mathbb{D}$?

2. The Legendre polynomials provide a complete, orthonormal basis for all functions on $x \in [-1, 1]$. We'll see them in action later, but in this problem we will study how they can be used to construct the derivative operator for functions on this domain.

   There are an infinite number of Legendre polynomials, which is necessary because the number of possible functions on this domain is also infinite. However, in this problem we'll just use the first three Legendre polynomials to construct part of the matrix that represents the derivative operator $\partial/\partial x$ on the domain $x \in [-1, 1]$. These polynomials are (up to normalization):

$$P_0(x) = \frac{1}{\sqrt{2}}\,, \tag{3}$$

$$P_1(x) = \sqrt{\frac{3}{2}}x\,, \tag{4}$$

$$P_2(x) = \sqrt{\frac{5}{8}}(3x^2 - 1)\,. \tag{5}$$

   (a) Verify that these Legendre polynomials are all orthonormal on the domain $x \in [-1, 1]$. That is, they are L^2-normalized and the "dot product" of two distinct Legendre polynomials is 0:

$$\int_{-1}^{1} dx\, P_m(x)\, P_n(x) = \delta_{mn}\,. \tag{6}$$

   (b) Using this basis of functions on $x \in [-1, 1]$, construct the 3×3 matrix that represents the derivative operator, $\partial/\partial x$. For example, the element in the first row and second column of the $\partial/\partial x$ matrix would be:

$$\left(\frac{\partial}{\partial x}\right)_{12} = \int_{-1}^{1} dx\, P_0(x) \frac{\partial}{\partial x} P_1(x)\,. \tag{7}$$

   Note the different indexing of the Legendre polynomials and the label of a row or column.

   (c) From this 3×3 derivative matrix, calculate its eigenvalues and eigenvectors. What does this result mean? Does the derivative $\partial/\partial x$ have eigenfunctions on the domain $x \in [-1, 1]$?

   (d) With only the first three Legendre polynomials, $P_0(x)$, $P_1(x)$, and $P_2(x)$, this is only a complete, orthonormal basis for general quadratic polynomials on $x \in [-1, 1]$. For a polynomial expressed as:

$$p(x) = ax^2 + bx + c\,, \tag{8}$$

   for some constants a, b, c, re-write it as a linear combination of the Legendre polynomials. Express the coefficients of this linear combination as a three-dimensional vector.

(e) Act on this three-dimensional vector with the derivative matrix that you constructed in part (b). Remember, the result represents another linear combination of Legendre polynomials. Does the result agree with what you would find from just differentiating the polynomial in part (d)?