Problem Set 10

Phys 342

Due April 17

Exercises, Due Friday, April 17
Email your homework to me at larkoski@reed.edu

1. Our analysis of the hydrogen atom simply extends to any element which has been ionized to have a single electron orbiting the nucleus. In this problem, we will consider such an atom, whose nucleus consists of Z protons and a single electron.

   (a) What are the energy eigenstates of this ionized atom? What is its "Bohr radius" and how does it compare to that of hydrogen?

   (b) What is the minimal value of Z such that the speed of the orbiting electron in the ground state is comparable to the speed of light, say, $v \sim c/2$? What element does this correspond to?

2. When studying the hydrogen atom, or other central potential problems like gravitation, you probably only exploited the conserved quantities of linear momentum, angular momentum, and energy. Linear momentum conservation means that there are no external forces in your problem, so you can work in the frame in which the center-of-mass is at rest. Angular momentum conservation means that there are no external torques, so for a central potential problem, you can restrict the analysis to a plane. Finally, energy conservation means that there are no non-conservative forces, so the energy at any time is determined by the initial configuration.

   However, in a central potential problem like that of gravity or electromagnetism with a potential that goes like $1/r$, there is another conserved quantity, called the **Laplace-Runge-Lenz vector** $\vec{A}$. In classical mechanics for electromagnetism, this vector is defined as

   $$\vec{A} = \vec{p} \times \vec{L} - \frac{m}{4\pi\epsilon_0}\hat{r}\,, \tag{1}$$

   where $\vec{p}$ is the momentum, $\vec{L}$ is the angular momentum, and $\hat{r}$ is the radial unit vector (i.e., the direction of the electric force). Quantum mechanically for the hydrogen atom, the ith component of the Laplace-Runge-Lenz vector can be defined as

   $$\hat{A}_i = -\frac{m_e}{4\pi\epsilon_0}\hat{r}_i + \sum_{\substack{j=1\\k=1}}^{3} \epsilon_{ijk}(\hat{p}_j\hat{L}_k + \hat{L}_k\hat{p}_j)\,. \tag{2}$$

Here, as earlier, $\hat{r}_i$ is the ith component of the unit radial vector, $r_i/|\vec{r}|$. As a conserved operator, it must commute with the Hamiltonian of the hydrogen atom, $[\hat{A}_i, \hat{H}] = 0$. While I encourage you to show this, it is quite detailed and beyond the scope of this homework. In this problem, we will calculate various commutators of this new vector and see what makes it so special.

(a) Calculate the commutator of the Laplace-Runge-Lenz vector with angular momentum. Show that

$$[\hat{A}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{A}_k \,. \tag{3}$$

To do this, it's enough for you to show that

$$[\hat{A}_z, \hat{L}_z] = 0 \,, \qquad\qquad [\hat{A}_x, \hat{L}_y] = i\hbar\hat{A}_z \,. \tag{4}$$

(b) The real power of the Laplace-Runge-Lenz vector is that it relates states with different values of squared angular momentum L^2. That is, it relates eigenstates of the electron in the hydrogen atom that live in different *representations* of rotations. Evaluate the commutator of the squared angular momentum operator L^2 and $\hat{A}_i$, $[\hat{A}_i, L^2]$. Simplify it as much as possible. In the simplification, arrange the result with all angular momentum operators as far to the right as possible. Does this vanish? Can you say what this means physically?

(c) Now, construct the raising, lowering, and z-component of the Laplace-Runge-Lenz vector, where

$$\hat{A}_+ = \hat{A}_x + i\hat{A}_y \,, \qquad\qquad A_- = \hat{A}_x - i\hat{A}_y \,. \tag{5}$$

Consider a state with $\ell = 0$, so also $m = 0$. Act $\hat{A}_+$ on this state; what does it return? That is, determine ℓ' and m' such that

$$\hat{A}_+|0,0\rangle = |\ell', m'\rangle \,. \tag{6}$$

Don't worry about overall normalization. Do the same thing for the states $\hat{A}_-|0,0\rangle$ and $A_z|0,0\rangle$; that is, what are ℓ'', m'', ℓ''', and m''' such that

$$\hat{A}_-|0,0\rangle = |\ell'', m''\rangle \,, \qquad\qquad \hat{A}_z|0,0\rangle = |\ell''', m'''\rangle \,. \tag{7}$$