Problem Set 2

Phys 342

Due February 14

Exercises, Due Friday, February 14

1. The Pauli matrices σ_1, σ_2, σ_3 form a complete basis for all 2×2 Hermitian matrices. The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

   In class, we studied the properties of σ_2 and σ_3.

   (a) Find the eigenvectors for all of the Pauli matrices. Enforce a normalization for the eigenvectors by demanding that their length is 1. That is, for an eigenvector $\vec{v}$, require that $\vec{v}^\dagger \vec{v} = 1$.

   (b) As emphasized in class, any explicit representation of a matrix requires a basis in which to write that matrix. Using the basis of eigenvectors of σ_1, re-write the three Pauli matrices of Eq. 1 above in that basis. What do you notice about the result?

   (c) As Hermitian matrices, they can be exponentiated to construct a corresponding unitary matrix. Let's exponentiate σ_3 as defined in Eq. 1 to construct the matrix

$$\mathbb{A} = e^{i\phi\sigma_3}, \tag{2}$$

   where ϕ is a real number. What is the resulting matrix $\mathbb{A}$? Write it in standard 2×2 form. Is it actually unitary?

2. The unitary matrix that implements rotations on two-dimensional vectors can be written as:

$$\mathbb{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3}$$

   where θ is the real-valued rotation angle.

   (a) Verify that this matrix is indeed unitary.

(b) As a unitary matrix, it can be expressed as the exponential of a Pauli matrix in the form

$$\mathbb{M} = e^{i\theta\sigma_j}\,, \tag{4}$$

for some Pauli matrix σ_j. Which Pauli matrix is it? Be sure to show your justification.

3. We introduced unitary operators as those that map the Hilbert space to itself; i.e., those that maintain the normalization constraint that we require of all vectors in the Hilbert space. However, we didn't show the converse, that there always exists a unitary operator that connects two vectors of the Hilbert space. For a Hilbert space $\mathcal{H}$ of two-dimensional vectors, consider the vectors $\vec{u}, \vec{v} \in \mathcal{H}$.

(a) We can express these vectors in a particular basis as

$$\vec{u} = \left(\begin{array}{c} e^{i\phi_1}\cos u \\ e^{i\phi_2}\sin u \end{array}\right)\,, \qquad \vec{v} = \left(\begin{array}{c} e^{i\theta_1}\cos v \\ e^{i\theta_2}\sin v \end{array}\right)\,, \tag{5}$$

for real parameters $u, v, \phi_1, \phi_2, \theta_1, \theta_2$. Show that these vectors are normalized to live in the Hilbert space.

(b) In this basis, determine the unitary matrix $\mathbb{U}$ such that

$$\vec{v} = \mathbb{U}\vec{u}\,. \tag{6}$$

Hint: Write the matrix $\mathbb{U}$ as

$$\mathbb{U} = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\,, \tag{7}$$

and determine the constraints on a, b, c, d if $\mathbb{U}$ is unitary and maps $\vec{u}$ to $\vec{v}$.

(c) Can the matrix $\mathbb{U}$ be degenerate? That is, can its determinant ever be 0?