Problem Set 3

Phys 342

Due February 21

Exercises, Due Friday, February 21
TURN IN TO MARY SULLIVAN BY 10 AM!!!

1. In lecture, we introduced completeness as the requirement that an orthonormal basis $\{|v_i\rangle\}_i$ satisfies
$$\sum_i |v_i\rangle\langle v_i| = \mathbb{I}\,. \tag{1}$$
In this problem, we will study different aspects of this completeness relation.

   (a) What if the basis is not orthogonal? Consider the vectors
$$\vec{v}_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)\,, \qquad \vec{v}_2 = \left(\begin{array}{c} e^{i\phi_1}\sin\theta \\ e^{i\phi_2}\cos\theta \end{array}\right)\,. \tag{2}$$
What is their outer product sum $|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$? For what value of θ does it satisfy the completeness relation? Does the result depend on ϕ_1 or ϕ_2?

   (b) In the first homework, we studied the first three Legendre polynomials as a complete basis for all quadratic functions on $x \in [-1,1]$. There we had demonstrated orthonormality:
$$\int_{-1}^{1} dx\, P_i(x)P_j(x) = \delta_{ij}\,. \tag{3}$$
What about completeness? First, construct the "identity matrix" $\mathbb{I}$ formed from the outer product of the first three Legendre polynomials:
$$\mathbb{I} = P_0(x)P_0(y) + P_1(x)P_1(y) + P_2(x)P_2(y)\,. \tag{4}$$
Use the form of the Legendre polynomials presented in Homework 1.

   (c) You should find something that does not look like an identity matrix. How can we test it? Provide an interpretation of this outer product identity matrix by integrating against an arbitrary quadratic function:
$$\int_{-1}^{1} dx\,(ax^2 + bx + c)\,\mathbb{I}\,, \tag{5}$$
for some constants a, b, c. What should you find? What do you find?

2. In this problem, we'll study the time dependence of a simple two-state system. Let's assume that $|1\rangle$ and $|2\rangle$ are energy eigenstates with energies E_1 and E_2, respectively. These states are orthonormal: $\langle i|j\rangle = \delta_{ij}$. We'll study the initial state at $t = 0$

$$|\psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle\,, \tag{6}$$

where α_1, α_2 are some complex coefficients such that $|\psi\rangle$ is in the Hilbert space.

(a) Let's give the state $|\psi\rangle$ time dependence. What is the state $|\psi(t)\rangle$, for general t?

(b) With this time-dependent state, compute the outer product $|\psi(t)\rangle\langle\psi(t)|$ for generic t, in the basis of energy eigenstates. Express the resulting matrix both in bra-ket notation and in familiar matrix notation. What do the diagonal elements represent?

(c) What is the Hamiltonian $\hat{H}$ in bra-ket notation, expressed in the $|1\rangle, |2\rangle$ basis? Is it Hermitian?

(d) Calculate the expectation value of the Hamiltonian $\hat{H}$ in the time-dependent state $|\psi(t)\rangle$, $\langle\psi(t)|\hat{H}|\psi(t)\rangle$. Does it depend on time?

(e) Let's assume that there is some other Hermitian operator on this Hilbert space, called $\hat{\mathcal{O}}$. We know its action on the energy eigenstates:

$$\hat{\mathcal{O}}|1\rangle = |1\rangle - |2\rangle\,, \qquad \hat{\mathcal{O}}|2\rangle = -|1\rangle + |2\rangle\,. \tag{7}$$

Express $\hat{\mathcal{O}}$ in both bra-ket notation and familiar matrix notation. Is it actually Hermitian?

(f) What is the expectation value of the unitary operator $\hat{\mathcal{O}}$ from part (e), $\langle\psi(t)|\hat{\mathcal{O}}|\psi(t)\rangle$? Does it depend on time? What does it simplify to if both α_1 and α_2 are real?

3. Neutrinos are very low mass, extremely weakly-interacting particles that permeate the universe. About a quadrillion passed through you while you read this. There are multiple types, or flavors, of neutrinos, and they can oscillate into one another as time passes. A model for the oscillations of neutrinos is the following. Consider two neutrinos that are also energy eigenstates, $|\nu_1\rangle$ and $|\nu_2\rangle$, with energies E_1 and E_2, respectively. Neutrinos are produced and detected not as energy eigenstates, but as eigenstates of a different Hermitian operator, called the flavor operator. There are two flavors of neutrino, called electron $|\nu_e\rangle$ and muon $|\nu_\mu\rangle$ neutrinos, and these can be expressed as a linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$ as:

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle\,, \qquad |\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle\,, \tag{8}$$

for some mixing angle θ. These flavor-basis neutrinos then travel for time T until they hit a detector where their flavor composition is measured.[1]

[1]Why and how neutrinos oscillate is actually quite subtle, and requires quantum mechanical and special relativistic considerations. See chapter 12 of A. J. Larkoski, "Elementary Particle Physics," Cambridge University Press (2019).

(a) Assume that the initial neutrino flavor is exclusively electron, $|\nu_e\rangle$. What is the electron-neutrino state after time T, $|\nu_e(T)\rangle$?

(b) After time T, what is the probability for the detector to measure an electron neutrino? What about a muon neutrino? The detector only measures those flavor eigenstates as defined by equation 8. You should find that the probability to measure the muon neutrino is

$$P_\mu = \sin^2(2\theta)\sin^2\left(\frac{(E_1 - E_2)T}{2\hbar}\right) . \tag{9}$$

Hint: The flavor basis is orthogonal and complete, so it can be used to express the identity operator.

(c) Describe why this phenomena is called neutrino *oscillations*.