Problem Set 4

Phys 342

Due February 28

Exercises, Due Friday, February 28

1. Consider the following operators that correspond to exponentiating the momentum and position operators, $\hat{x}$ and $\hat{p}$:

$$\mathbb{X} = e^{i\sqrt{\frac{2\pi c}{\hbar}}\hat{x}}\,, \qquad \mathbb{P} = e^{i\sqrt{\frac{2\pi}{c\hbar}}\hat{p}}\,. \tag{1}$$

   Here, c is some constant.

   (a) What are the units of the constant c? From that, can you provide an interpretation of c?

   (b) Calculate the commutator of $\mathbb{X}$ and $\mathbb{P}$, $[\mathbb{X}, \mathbb{P}]$. What does this mean?

   (c) What is the range of eigenvalues of $\hat{x}$ and $\hat{p}$ for which the operators $\mathbb{X}$ and $\mathbb{P}$ are single-valued? Remember, $e^{i\pi} = e^{i3\pi}$, for example.

   (d) Determine the eigenstates and eigenvalues of the exponentiated momentum operator, $\mathbb{P}$. That is, what values can λ take and what function $f_\lambda(x)$ satisfies

$$\mathbb{P} f_\lambda(x) = \lambda f_\lambda(x)\,? \tag{2}$$

   Do the eigenvalues λ have to be real valued? Why or why not?

   (e) Using part (b), what can you say about the eigenstates of $\mathbb{X}$?

2. Consider two Hermitian operators $\hat{A}$ and $\hat{B}$ and consider the unitary operators formed from exponentiating them:

$$\mathbb{U}_A = e^{i\hat{A}}\,, \qquad \mathbb{U}_B = e^{i\hat{B}}\,. \tag{3}$$

   If $\hat{A}$ and $\hat{B}$ were just numbers, then it would be easy to determine the unitary matrix formed from the product of $\mathbb{U}_A$ and $\mathbb{U}_B$. However, matrices do not in general commute, so it complicates this product. Using the Taylor expansion of the exponential, find the difference of unitary matrices

$$e^{i\hat{A}}e^{i\hat{B}} - e^{i(\hat{A}+\hat{B})}\,. \tag{4}$$

   Only consider terms in the Taylor expansion up through cubic order; that is, terms that contain at most the product of three matrices $\hat{A}$ and/or $\hat{B}$. Under what condition does this difference vanish?

3. An **instanton** is a quantum mechanical excitation that is localized in space like a particle. They are closely related to solitary waves or solitons that were first observed in the mid-19th century as a traveling wave in a canal. We'll study a model for instantons in this problem. Our simple model will be the following. We will consider a quantum system constrained on a circle, and we can define states on this circle by their **winding number** n, the number of times that the instanton wraps around the circle before it connects back to itself (think about winding a string around a cylinder and then tying it back together after n times around). The winding number n can be any integer, positive, negative, or zero, and the sign of the winding number encodes the direction in which it is wrapped.

   States with different n are orthogonal, so we will consider the Hilbert space as spanned by the set of states $\{|n\rangle\}_{n=-\infty}^{\infty}$ which are orthonormal:

$$\langle m|n\rangle = \delta_{mn} \,, \tag{5}$$

   and we will assume they are complete.

   (a) On this Hilbert space, we can define a **hopping operator** $\mathcal{O}$ which is defined to act on the basis elements as:

$$\mathcal{O}|n\rangle = |n+1\rangle \,. \tag{6}$$

   Show that this means that $\mathcal{O}$ is unitary.

   (b) Assume that the state $|\psi\rangle$ is an eigenstate of $\mathcal{O}$ with eigenvalue defined by an angle θ:

$$\mathcal{O}|\psi\rangle = e^{i\theta}|\psi\rangle \,. \tag{7}$$

   Express the state $|\psi\rangle$ as a linear combination of the winding states $|n\rangle$.

   (c) The Hamiltonian for this winding system $\hat{H}$ is defined to act as

$$\hat{H}|n\rangle = |n|E_0|n\rangle \,, \tag{8}$$

   where E_0 is a fixed energy and $|n|$ is the absolute value of the winding number n. Calculate the commutator of the hopping operator and the Hamiltonian, $[\hat{H}, \mathcal{O}]$.

   (d) Determine the time dependence of the state $|\psi\rangle$; that is, evaluate

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi\rangle \,. \tag{9}$$