Problem Set 5

Phys 342

Due March 6

Exercises, Due Friday, March 6

1. Is the momentum operator $\hat{p}$ in the infinite square well Hermitian? Prove that it is or provide a counterexample.

2. In lecture, we discussed how the energy ground state of the infinite square well, $|\psi_1\rangle$, was the state that minimized the uncertainty relation. That is, in this state, the product of the the variance of the momentum and position $\sigma_x^2\sigma_p^2$, was minimized. Further, we demonstrated that for all energy eigenstates

$$\sigma_x^2\sigma_p^2 \geq \frac{\hbar^2}{4}\,, \tag{1}$$

as required. However, consider the momentum eigenstate

$$\psi(x) = \frac{e^{i\frac{px}{\hbar}}}{\sqrt{a}}\,, \tag{2}$$

for some momentum p, in the infinite square well. Verify that it is indeed L^2-normalized on the well and calculate both variances σ_x^2 and σ_p^2. Is the uncertainty principle satisfied? Why or why not?

3. In this problem, we will work to understand the time evolution of wavefunctions that are localized in position in the infinite square well.

   (a) It will prove simplest later to re-write the infinite square well in a way that is symmetric for $x \to -x$. So, for an infinite square well in which the potential is 0 for $x \in [-\pi/2, \pi/2]$, find the energy eigenstates $\psi_n(x)$ and the corresponding energy eigenvalues E_n.

   (b) Now, let's consider the initial wavefunction $\psi(x)$ that is a uniform bump in the middle of the well:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\,, & |x| < \pi/4\,, \\ 0\,, & |x| > \pi/4\,. \end{cases} \tag{3}$$

   From this initial wavefunction, determine the wavefunction at a general time t, $\psi(x,t)$.

(c) Does this wavefunction "leak" into the regions where it was initially 0? Let's take the inner product of this time dependent wavefunction with the wavefunction $\chi(x)$ that is uniform over the well:

$$\chi(x) = \frac{1}{\sqrt{\pi}} \,. \tag{4}$$

What is $\langle \chi | \psi \rangle$, as a function of time?

(d) What is the first time derivative of the inner product at $t = 0$,

$$\left. \frac{d\langle \chi | \psi \rangle}{dt} \right|_{t=0} ? \tag{5}$$

Can you think about what this means in the context of the Schrödinger equation?

(e) On the state $\psi(x,t)$, what are the expectation values of position and momentum for all time, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$?

4. We like to think of the $E \to \infty$ limit as the limit in which quantum mechanics "turns into" classical mechanics, but this clearly has limitations. The limitation that we will consider here is the fact that if the energy density of a particle's wavefunction is too high, then it will create a black hole. For a total energy E, a black hole is created if this is packed into a region smaller than its Schwarzschild radius, R_s. For energy E, the Schwarzschild radius is

$$R_s = \frac{2G_N E}{c^4} \,, \tag{6}$$

where G_N is Newton's constant and c is the speed of light.

(a) For a general energy eigenstate $|\psi_n\rangle$ of the infinite square well, determine its Schwarzschild radius. For what value of energy level n does the Schwarzschild radius equal the size a of the infinite square well? In this part, you can leave the answer in terms of the constants provided in the problem.

(b) What is the energy E_n for which the size of the well is the Schwarzschild radius? Evaluate this for a well that's the size of an atomic nucleus, $a = 10^{-15}$ m. Compare this energy to some "everyday" object's energy (something like kinetic energy of a baseball, energy of photons from the sun, etc.).

(c) Using part (a), what energy eigenstate level n does this correspond to? Take the mass m of the object to be that of the pion, a subatomic particle responsible for binding atomic nuclei. The mass of the pion m_π is:

$$m_\pi = 2.4 \times 10^{-28}\ \text{kg} \,. \tag{7}$$

How does this compare to the energy level n you would predict that the pion would be traveling at the speed of light, c? You can use the approximation that $\hbar = 10^{-34}\ \text{J}\cdot\text{s}$.