Problem Set 6

Phys 342

Due March 13

Exercises, Due Friday, March 13

1. In class, we studied coherent states, which we had identified as eigenstates of the lowering operator, a. A coherent state $|\psi\rangle$ is defined by the eigenvalue equation

$$a|\psi\rangle = \lambda|\psi\rangle\,, \tag{1}$$

where λ is a complex number. In terms of the harmonic oscillator's ground state $|\psi_0\rangle$, this coherent state can be expressed as

$$|\psi\rangle = \beta_0 e^{\lambda a^\dagger}|\psi_0\rangle\,, \tag{2}$$

where β_0 is some normalization constant. In this problem, we will study the properties of these coherent states.

(a) First, determine the normalization constant β_0 such that $\langle\psi|\psi\rangle = 1$.

(b) We would like to determine the time evolution of this state. To do so, we need to act with the exponentiated Hamiltonian; that is,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi\rangle = \beta_0 e^{-i\frac{\hat{H}t}{\hbar}} e^{\lambda a^\dagger}|\psi_0\rangle\,. \tag{3}$$

The first step to evaluate this is to determine the commutator of the exponentiated Hamiltonian and raising operator:

$$\left[e^{-i\frac{\hat{H}t}{\hbar}}, e^{\lambda a^\dagger}\right]\,. \tag{4}$$

What is this?

(c) Using the result of part (b), write down the time-dependent coherent state $|\psi(t)\rangle$. What is the action of the lowering operator a on this time-evolved state,

$$a|\psi(t)\rangle\,? \tag{5}$$

(d) Calculate the variances of the position and momentum operators $\hat{x}$ and $\hat{p}$ on this time-evolved state $|\psi(t)\rangle$. What is the Heisenberg uncertainty principle now?

(e) Why don't we consider eigenstates of the raising operator, $a^\dagger$? What is wrong with a state $|\chi\rangle$ that satisfies

$$a^\dagger|\chi\rangle = \eta|\chi\rangle\,, \tag{6}$$

for some complex number η?

2. **Supersymmetry** is a proposed extension of the symmetries of spacetime beyond that of just familiar Lorentz transformations and translations. A supersymmetry transformation interchanges fermions and bosons, which are particles of different spins and correspondingly different statistical properties. While there is no evidence for supersymmetry in Nature, it is nevertheless an interesting proposal that has produced new insights into familiar phenomena.

   Supersymmetric quantum mechanics brings some of the ideas of the particle physics supersymmetry to non-relativistic quantum mechanics. In this problem, we will study features of this formulation of quantum mechanics. The first step is to identify two operators A and $A^\dagger$ such that

$$A^\dagger A = \hat{H} - E_0\,, \tag{7}$$

   where $\hat{H}$ is the familiar Hamiltonian and E_0 is its ground state energy (i.e., just a number). A and $A^\dagger$ are Hermitian conjugates of one another and in general can be expressed as:

$$A = \frac{i}{\sqrt{2m}}\hat{p} + W(\hat{x})\,, \qquad\qquad A^\dagger = -\frac{i}{\sqrt{2m}}\hat{p} + W(\hat{x})\,, \tag{8}$$

   where $\hat{p}$ and $\hat{x}$ are the familiar momentum and position operators. Here, $W(\hat{x})$ is called the superpotential and is a real function of $\hat{x}$.

   (a) With these relationships, determine the potential $V(\hat{x})$ in terms of the superpotential $W(\hat{x})$.

   (b) With this definition of A and $A^\dagger$, what is their commutator, $[A, A^\dagger]$?

   (c) We can instead consider the *anti-commutator* $\{A, A^\dagger\}$ defined as

$$\{A, A^\dagger\} = AA^\dagger + A^\dagger A\,. \tag{9}$$

   What is this?

   The anti-commutator is present in the algebra that defines supersymmetry and its nice properties with these definitions is why this is called "supersymmetric quantum mechanics." The algebra of supersymmetry consists of both commutation and anti-commutation relations and is referred to as a "graded Lie algebra." We'll discuss Lie algebras in great detail starting in a couple of weeks.

   (d) Now, connecting back to the harmonic oscillator, what is its superpotential? When we constructed the raising and lowering operators a and $a^\dagger$ in class, we just used their commutator $[a, a^\dagger]$. Why could we get away with that, and didn't need to use their anti-commutator?

   (e) *Not for credit*: What is the superpotential for the infinite square well?