Problem Set 7

Phys 342

Due March 30

Exercises, Due Monday, March 30
Email your homework to me at larkoski@reed.edu

1. Let's first consider a free-particle, whose wavefunction can be expressed as

$$\psi(x,t) = \int_{-\infty}^{\infty} dp\, g(p)\, e^{-i\frac{E_p t - px}{\hbar}}\,, \tag{1}$$

where $g(p)$ is a complex-valued, L^2-normalized function of momentum p and $E_p = p^2/2m$, the kinetic energy.

   (a) Assume that this wavefunction is a coherent state at time $t = 0$:

$$a|\psi\rangle = \lambda|\psi\rangle\,, \tag{2}$$

   where a is the lowering operator we introduced with the harmonic oscillator and λ is a complex number. What differential equation must $g(p)$ satisfy for this wavefunction to be a coherent state?

   (b) What is the speed of center-of-probability of this initial coherent state, at time $t = 0$? What is the acceleration of the center-of-probability for any time t? From these results, provide an interpretation of λ.

   (c) We had demonstrated that the eigenstates of $a^\dagger$ had problems in the harmonic oscillator. For the free-particle, what are the states that are eigenstates of the raising operator, $a^\dagger$? Are they allowed in this case?

2. For this problem, you will use properties of the reflection A_R and transmission A_T amplitudes that we derived in class:

$$A_R = \frac{mV_0}{k^2 - mV_0 + ik\sqrt{k^2 - 2mV_0}\cot\left(\frac{a}{\hbar}\sqrt{k^2 - 2mV_0}\right)}\,, \tag{3}$$

$$A_T = \frac{ke^{-i\frac{ka}{\hbar}}\sqrt{k^2 - 2mV_0}}{k\sqrt{k^2 - 2mV_0}\cos\left(\frac{a}{\hbar}\sqrt{k^2 - 2mV_0}\right) - i(k^2 - mV_0)\sin\left(\frac{a}{\hbar}\sqrt{k^2 - 2mV_0}\right)}\,. \tag{4}$$

We'll explore the properties of these amplitudes for complex-valued momentum $k \in \mathbb{C}$. In particular, there is a very general property of the S-matrix that poles (divergences) in the complex momentum plane correspond to bound states of the system. After completing this problem, read through Section 2.6 of Griffiths and Schroeter.

(a) Show that the poles of the reflection and transmission amplitudes are located at the same value of k.

(b) Assume that the potential is weak and narrow: that is, the quantity

$$\frac{a}{\hbar}\sqrt{2mV_0} \to 0\,. \tag{5}$$

In this limit, are there any poles of the transmission and reflection amplitudes? How many are there? At what values of k are they located?
Hint: Be sure to work in the $V_0 \to 0$ and $a \to 0$ limits throughout. If $V_0 = 0$, where must the pole(s) be located in k?

(c) Continuing working in this shallow and narrow limit, we can express the reflection and transmission amplitudes as a sum over these poles, where

$$A = \sum_{\text{poles } k_n} \frac{c_n(k)}{k - k_n}\,, \tag{6}$$

where A is the amplitude (reflection or transmission), $c_n(k)$ is a polynomial function of k, and k_n is the location of the nth pole. Write the reflection and transmission amplitudes in this form. Can you provide some physical meaning to the numerators $c_n(k)$, called the residues?
Hint: How does k^2 compare to mV_0 in this limit?

(d) Consider the potential

$$V(x) = \begin{cases} 0\,, & x < 0\,,\ x > a\,, \\ \frac{V_0 l}{a}\,, & 0 < x < a\,, \end{cases} \tag{7}$$

where l is a fixed length scale. What are the reflection and transmission amplitudes in the limit that $a \to 0$?

(e) How many poles are there in the reflection and transmission amplitudes in this $a \to 0$ limit? At what values of k are they located?

(f) The sum of the reflection and transmission amplitudes represents the total amplitude for an outgoing wave scattered off of the potential. Further, conservation of probability requires that the total incident probability equals the total outgoing probability, so we can express the sum of the reflection and transmission amplitudes as a complex exponential, in general. That is,

$$A_R + A_T = e^{2i\delta}\,, \tag{8}$$

where δ is real. For the potential in the limit of part (d), what is δ if $k = 0$? What about $k \to \infty$? Can you provide a physical interpretation of δ?