Problem Set 8

Phys 342

Due April 3

Exercises, Due Friday, April 3
Email your homework to me at larkoski@reed.edu

1. The Jacobi identity is a requirement of the commutation relations of a Lie algebra that ensures that the corresponding Lie group is associative. For elements A, B, C of a Lie algebra, the Jacobi identity is:

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0\,. \tag{1}$$

In this expression, $[A, B]$ is called the Lie bracket of the Lie algebra. We have only studied Lie brackets that correspond to the familiar commutator, but it is possible to consider other definitions that satisfy the Jacobi identity.

(a) Show that the Jacobi identity is satisfied for the Lie algebra of the three-dimensional rotation group, with Lie bracket

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k\,. \tag{2}$$

(b) Show that the Jacobi identity is satisfied for any Lie algebra for which the Lie bracket is the just the commutator, that is, for

$$[A, B] = AB - BA\,. \tag{3}$$

2. In lecture, we showed that three-dimensional rotations are not commutative, and this has consequences for how rotations are applied on an object. In this problem, we will consider the two-dimensional representation of the rotation group and its action on spinors.

(a) First, construct the unitary operators and corresponding 2×2 matrices that implement a rotation by angle θ about either the x or y axes. That is, what are the matrices

$$\mathbb{U}_x(\theta) = e^{i\frac{\theta\hat{S}_x}{\hbar}} \qquad \text{and} \qquad \mathbb{U}_y(\theta) = e^{i\frac{\theta\hat{S}_y}{\hbar}}\ ? \tag{4}$$

(b) What is the commutation relation of these matrices; that is, what is

$$[\mathbb{U}_x(\theta), \mathbb{U}_y(\theta)]\,? \tag{5}$$

For what values of θ does this vanish?

(c) Consider the action of $\mathbb{U}_x(\theta)$ on an eigenstate of $\hat{S}_z$. In particular, evaluate

$$\mathbb{U}_x(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6}$$

Are there values of θ for which this is transformed into an eigenstate of $\hat{S}_y$?

(d) In class, we explicitly compared what happens to a three-dimensional object when it is rotated about two distinct axes by 90° in different orders. First, draw a picture of how an initial three-dimensional vector $\vec{v} = (0\;\;0\;\;1)^\intercal = \hat{z}$ is rotated in two ways: first 90° about the x axis, and then 90° about the y axis, and then vice-versa. Call these resulting vectors $\vec{v}_{xy}$ and $\vec{v}_{yx}$, respectively. Note that the initial vector $\vec{v}$ is just the unit vector along the z axis. Your drawing should show three vectors: the initial vector $\vec{v}$, and the two final vectors corresponding to action of the rotation in two different orders. What is the difference of the resulting vectors, $\vec{v}_{xy} - \vec{v}_{yx}$?

(e) Now, act the commutator $[\mathbb{U}_x(\theta), \mathbb{U}_y(\theta)]$ with $\theta = 90°$ on the eigenstate of $\hat{S}_z$ from part (c). What is

$$\left[\mathbb{U}_x\left(\frac{\pi}{2}\right), \mathbb{U}_y\left(\frac{\pi}{2}\right)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}? \tag{7}$$

Is this equivalent to the difference of vectors $\vec{v}_{xy} - \vec{v}_{yx}$ you found in part (d)? To answer this problem, you have to think about how vectors in different representations of the rotation group can correspond to one another.
Hint: Remember, a minus sign $-$ is just a rotation by 180° for a three-dimensional vector. What is a 180° rotation on a spinor?