Problem Set 9

Phys 342

Due April 10

Exercises, Due Friday, April 10
Email your homework to me at larkoski@reed.edu

1. The Killing form of a Lie algebra provides the definition of normalization of operators in a particular representation of the Lie algebra. For representation R of $\mathfrak{su}(2)$, the Killing form is
$$\text{tr}\left[\hat{L}_i^{(R)}\hat{L}_j^{(R)}\right] = k_R\,\delta_{ij}\,, \tag{1}$$
where tr denotes the trace (i.e., the sum of diagonal elements). The quantity k_R depends on the representation. For a representation R of dimension D and Casimir C_R, where
$$C_R\mathbb{I}_D = \left(\hat{L}_x^{(R)}\right)^2 + \left(\hat{L}_y^{(R)}\right)^2 + \left(\hat{L}_z^{(R)}\right)^2\,, \tag{2}$$
and $\mathbb{I}_D$ is the $D\times D$ identity matrix, express k_R in terms of D and C_R. What is k_R for a representation of spin ℓ?

2. We've studied coherent states in the context of the harmonic oscillator and the free particle, and our formulation of angular momentum suggests that there is a definition of coherent state in that case, too. We constructed the angular momentum raising and lowering operators, $\hat{L}_+$ and $\hat{L}_-$, and we could in principle define a coherent state to be an eigenstate of L_-, for example.

   Consider an eigenstate of the angular momentum lowering operator for spin ℓ:
$$\hat{L}_-|\psi\rangle = \lambda|\psi\rangle\,. \tag{3}$$
   What is this state? What are the possible values of λ? Does it exist in the Hilbert space?

3. An ideal model of a two-state quantum system is of an electrically-charged, spin-1/2 particle immersed in a uniform magnetic field. The spin-1/2 particle therefore has a magnetic moment and that correspondingly interacts with the external magnetic field.

Let's consider a magnetic field $\vec{B} = B_0\hat{z}$ and let's put an electron in it. Then, the Hamiltonian of such an electron in the magnetic field is

$$\hat{H} = \frac{e\hbar}{2m_e}B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

where $e = 1.6\times10^{-19}$ C is the fundamental electric charge of the electron, and m_e is the mass of the electron. The first factor in the Hamiltonian is called the Bohr magneton and is simply the magnetic moment of the electron:

$$\mu_B = \frac{e\hbar}{2m_e}. \tag{5}$$

(a) What is the time evolution of the expectation value of the operator $\hat{S}_z$? What about $\hat{S}_x$ and $\hat{S}_y$? Provide a physical interpretation of what you find. Don't forget Ehrenfest's theorem.

(b) What is the uncertainty principle for energy and x-component of the spin of the electron? That is, for variances σ_E^2 and $\sigma_{S_x}^2$ for the energy and x-component of the spin, respectively, what is their minimum product:

$$\sigma_E^2\sigma_{S_x}^2 \geq ? \tag{6}$$

(c) When does the lower bound in the uncertainty relation vanish? Show that this makes sense from when we know the variances vanish.

4. A very powerful technique for expressing amplitudes of scattering processes in quantum field theory, the harmonious marriage of special relativity and quantum mechanics, is through **spinor helicity**, in which every quantity is related to the eigenstates of the spin-1/2 $\hat{S}_z$ operator. This is exceptionally convenient, because eigenstates of spin-1/2 are just two-component spinors! What could be simpler.

   In this problem, we will just study one identity that is often exploited in this business, called the Schouten identity. For four spin-1/2 states $|\psi\rangle, |\rho\rangle, |\chi\rangle, |\eta\rangle$, it states that

$$\langle\psi|\rho\rangle\langle\chi|\eta\rangle = \langle\psi|\eta\rangle\langle\chi|\rho\rangle + (\langle\rho|)^* i\sigma_2|\eta\rangle\langle\psi|i\sigma_2(|\chi\rangle)^*. \tag{7}$$

   Prove this equality. Note that the complex conjugation acts only on a single bra or ket in the final term.