Physics 342 Lecture 17

Welcome back to more analysis of the simple harmonic oscillator! Last lecture we had started studying this system for which its Hamiltonian is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

where a and $a^\dagger$ are Hermitian conjugates of one another:

$$(a)^\dagger = a^\dagger, \quad (a^\dagger)^\dagger = a, \text{ and}$$

their commutator is: $[a, a^\dagger] = 1$. Our goal in this lecture is to find the eigenvalues and eigenstates of this Hamiltonian; that is find the states $|\psi\rangle$ and energy E such that

$$\hat{H}|\psi\rangle = E|\psi\rangle.$$

We had learned last lecture that we <u>must</u> restrict E to be non-negative; otherwise if E is negative, then there is no minimum eigenvalue of $\hat{H}$. A physical system <u>must</u> have a minimum energy, so negative eigenvalues must be forbidden. With that restriction, then the smallest eigenvalue of the $a^\dagger a$ operator is 0; all other eigenvalues <u>must</u> be positive. So, let's just assume that $|\psi_0\rangle$ is the state for which it has 0 eigenvalue of $a^\dagger a$:

$$a^\dagger a|\psi_0\rangle = 0|\psi_0\rangle = 0.$$

By the way, we say that a state with 0 eigenvalue of some operator is annihilated by that operator. "Annihilated" properly means taken out of the Hilbert space; in particular "0" is not in Hilbert space, if for no other reason than it cannot have the correct, unitary, normalization. Such a state is also an eigenstate of the Hamiltonian:

$$\hat{H}|\psi_0\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|\psi_0\rangle = \frac{\hbar\omega}{2}|\psi_0\rangle.$$

As we have argued that $a^\dagger a$ can have no negative eigenvalues, this $|\psi_0\rangle$ state corresponds to the smallest eigenvalue of $\hat{H}$. We therefore call it the ground state, and the lowest energy of the Hamiltonian is:

$$E_0 = \frac{\hbar\omega}{2}.$$

This is fascinating. Consider a classical harmonic oscillator, like a mass on a spring. It is definitely possible for the mass to be at rest (with 0 kinetic energy) at the spring's equilibrium point (with 0 potential energy. Thus, such a classical system can have, simply, 0 energy. Quantum mechanically, however, this apparently can't be true: a quantum harmonic oscillator always has a non-zero energy, even in the ground state.

We had seen last lecture how to go from one value of energy to another, using the action of the a and $a^\dagger$ operators individually. So, let's see if we can leverage this ground state into constructing states with larger eigenvalues. Let's consider the state formed by acting $a^\dagger$ on $|\psi_0\rangle$:

$a^{+}|\psi_0\rangle$, and then the action of the Hamiltonian on it is:

$$\hat{H}\, a^{+}|\psi_0\rangle = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right) a^{+}|\psi_0\rangle$$

$$= \hbar\omega\left(a^{+}a^{+}a + a^{+}[a, a^{+}] + \tfrac{1}{2}a^{+}\right)|\psi_0\rangle$$

$$= \hbar\omega\left(1+\tfrac{1}{2}\right) a^{+}|\psi_0\rangle,$$

where I have used that $a|\psi_0\rangle = 0$. Then, apparently,

$\hat{H} a^{+}|\psi_0\rangle = \hbar\omega\left(1+\frac{1}{2}\right)a^{+}|\psi_0\rangle$, is an eigenstate with eigenvalue larger than the ground state energy by $\hbar\omega$.

This hints at a general construction of eigenvalues and eigenstates. Let's consider the state formed by acting on $|\psi_0\rangle$ with a^{+} n times:

$(a^{+})^n|\psi_0\rangle$. Then, the action of the Hamiltonian on such a state is: $\hat{H}(a^{+})^n|\psi_0\rangle = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)(a^{+})^n|\psi_0\rangle$

$$= \hbar\omega\left(\cancel{(a^{+})^{n+1}a} + a^{+}[a,(a^{+})^n] + \tfrac{1}{2}(a^{+})^n\right)|\psi_0\rangle$$

$$= \hbar\omega\left(a^{+}[a,(a^{+})^n] + \tfrac{1}{2}(a^{+})^n\right)|\psi_0\rangle$$

Last lecture, we had showed that the commutator:

$[a^{+}, a^n] = -n\,a^{n-1}$. If we Hermitian conjugate it, we have:

$$[a^{+}, a^n]^{+} = -n(a^{+})^{n-1} = [(a^{+})^n, a]$$

Note how the order of the commutator switches, as it is nothing more than a product of operators. So, we have:

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}.$$

Using this in our result above, we find:

$$\hat{H}(a^\dagger)^n|\psi_0\rangle = \hbar\omega\left((a^\dagger)^n n + \tfrac{1}{2}(a^\dagger)^n\right)|\psi_0\rangle = \hbar\omega(n+\tfrac{1}{2})(a^\dagger)^n|\psi_0\rangle.$$

So, through the action of $a^\dagger$, we can construct eigenstates of the Hamiltonian that have eigenvalue:

$$E_n = \hbar\omega(n+\tfrac{1}{2}), \text{ for the } n^{th} \text{ application of } a^\dagger.$$

We thus refer to the a, $a^\dagger$ operators as "ladder operators", as they can be used to go up and down the ladder of energy eigenvalues of $\hat{H}$.

While we note that $(a^\dagger)^n|\psi_0\rangle$ is an eigenstate of the Hamiltonian, it is only additionally in the Hilbert space if its norm is 1. Let's assume that $|\psi_0\rangle$ is normalized: $\langle\psi_0|\psi_0\rangle = 1$. What is the inner product of $(a^\dagger)^n|\psi_0\rangle$ with itself? Note that its Hermitian conjugate is:

$$\left((a^\dagger)^n|\psi_0\rangle\right)^\dagger = \langle\psi_0|a^n, \text{ so the inner product is:}$$

$$\langle\psi_0|a^n(a^\dagger)^n|\psi_0\rangle = \langle\psi_0|a^{n-1}(a^\dagger)^n a + a^{n-1}[a,(a^\dagger)^n]|\psi_0\rangle$$

$$= \langle\psi_0|a^{n-1}n(a^\dagger)^{n-1}|\psi_0\rangle,$$ where we have used the commutator of a and $(a^\dagger)^n$ from earlier.

So, ~~it~~ we ~~can~~ have the interesting recursion relation:

$$\langle \psi_0 | (a)^n (a^\dagger)^n | \psi_0 \rangle = n \langle \psi_0 | (a)^{n-1} (a^\dagger)^{n-1} | \psi_0 \rangle.$$

$$= n(n-1) \langle \psi_0 | a^{n-2} (a^\dagger)^{n-2} | \psi_0 \rangle$$
$$\vdots$$
$$= n! \langle \psi_0 | \psi_0 \rangle = n!$$

So, indeed, the state $(a^\dagger)^n |\psi_0\rangle$ is in general not in the Hilbert space. However, the fix is easy: we just need to include an appropriate normalization factor. We call:

$$|\psi_n\rangle \equiv \frac{(a^\dagger)^n}{\sqrt{n!}} |\psi_0\rangle,$$

the n^{th} energy eigenstate of the harmonic oscillator Hamiltonian. This state is normalized:

$$\langle \psi_n | \psi_n \rangle = \frac{\langle \psi_0 | a^n (a^\dagger)^n | \psi_0 \rangle}{n!} = 1$$ and has eigenvalue equal to $\hbar\omega(n+\frac{1}{2})$:

$$\hat{H} |\psi_n\rangle = \hbar\omega(a^\dagger a + \tfrac{1}{2}) (a^\dagger)^n |\psi_0\rangle = (n+\tfrac{1}{2})\hbar\omega |\psi_n\rangle.$$

So, simply from properties of the operators a, $a^\dagger$, we have very simply constructed an infinite tower/ladder of energy eigenstates and their eigenvalues. Note also that these states are orthogonal. For $n \neq m$, we have the inner product

$$\langle \psi_n | \psi_m \rangle = \frac{\langle \psi_0 | a^n (a^\dagger)^m | \psi_0 \rangle}{\sqrt{n!m!}}$$

Without loss of generality, we can assume that $n > m$.

Then, we have

$$\langle \psi_0 | a^{n-m} a^m (a^\dagger)^m | \psi_0 \rangle = \langle \psi_0 | a^{n-m} a^{m-1} \left((a^\dagger)^m a + [a, (a^\dagger)^m] \right) | \psi_0 \rangle$$

$$= \langle \psi_0 | a^{n-m} a^{m-1} m (a^\dagger)^{m-1} | \psi_0 \rangle$$

$$\vdots$$

$$= m! \langle \psi_0 | a^{n-m} | \psi_0 \rangle = 0, \text{ because } n-m > 0.$$

Thus, it appears that these energy eigenstates could be a basis for all states in the Hilbert space, as they are orthonormal. If they are to do this, they must further be complete. Recall that a basis of vectors is complete if, for basis $\{\vec{v}_i\}_{i=1}^N$:

$$\sum_{i=1}^N \vec{V}_i \vec{V}_i^T = \mathbb{I}_{N\times N}, \text{ the } N\times N \text{ identity matrix.}$$

In the case at hand, if these discrete energy eigenstates form a complete, orthonormal basis, we must have:

$$\sum_{n=0}^\infty |\psi_n\rangle\langle\psi_n| = \sum_{n=0}^\infty \frac{1}{n!} (a^\dagger)^n |\psi_0\rangle\langle\psi_0| a^n = \mathbb{I},$$

the identity operator. Actually proving this relationship is very subtle and essentially outside the realm of this class. However, we can take a pragmatic approach: let's assume that the $\{|\psi_n\rangle\}$ form a complete basis. Then, we can express an arbitrary state $|\psi\rangle$ as a linear combination of the $|\psi_n\rangle$:

$$|\psi\rangle = \sum_{i=0}^\infty \beta_n |\psi_n\rangle, \text{ and normalization requires that } \sum_{n=0}^\infty |\beta_n|^2 = 1.$$

If this basis is incomplete, then there will exist some state such that it cannot be expressed in this way. If we find such a state that's outside the Hilbert space spanned by $\{|\psi_n\rangle\}$, then we know it is incomplete and we have more work to do. However, if we never find such a state, then the completeness assumption is at least good enough for all of the states we consider. Further, if a state $|\psi\rangle$ is completely described by a linear combination of the $|\psi_n\rangle$ states, then, by construction, the Hamiltonian <u>only</u> acts on $|\psi\rangle$ to produce another state that is completely described by the $|\psi_n\rangle$. So, at least this assumption will be self-consistent for our applications.

As a final point for today, let's go back to position space and make sense of the ground state $|\psi_0\rangle$. This is annihilated by a:

$a|\psi_0\rangle = 0$, or, in terms of $\hat{p}$ and $\hat{x}$:

$$\left(i\frac{\hat{p}}{\sqrt{2m\hbar\omega}} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \right)|\psi_0\rangle = 0$$

Replacing $\hat{p}$ by the derivative operator, this becomes the differential equation:

$$\sqrt{\frac{\hbar}{2m\omega}}\frac{\partial}{\partial x}\psi_0(x) = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0(x), \text{ or}$$

$$\frac{\partial}{\partial x}\psi_0(x) = -\frac{m\omega}{\hbar}x\,\psi_0(x).$$

This can be solved to yield:

$$\psi_0(x) = N \exp\left[-\frac{m\omega}{2\hbar} x^2\right],$$ where N is a normalization constant.

This is fascinating: this function is the shape of a bell-curve, normal distribution, or Gaussian. What makes this Gaussian so special and why must the energy of the harmonic oscillator be at least $\hbar\omega/2$? More next time...