Physics 342 Lecture 18

Welcome to this Friday lecture! Please hand in homework.

This week we have been studying the harmonic oscillator and identified its energy eigenvalues and eigenstates. We had identified the Hamiltonian as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

$$\text{where } a = i\frac{\hat{p}}{\sqrt{2m\hbar\omega}} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}, \quad a^\dagger = -i\frac{\hat{p}}{\sqrt{2m\hbar\omega}} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}.$$

We call a and $a^\dagger$ "ladder operators" as they satisfy the commutation relation:

$$[a, a^\dagger] = 1$$

The energy eigenvalues of this Hamiltonian are

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where n is a non-negative integer: $n \in \{0, 1, 2, \ldots\}$. We call $E_0 = \frac{\hbar\omega}{2}$ the <u>ground state energy</u>. The eigenstate of the ground state can be denoted as $|\psi_0\rangle$, and we demonstrated that it is <u>annihilated</u> by a:

$a|\psi_0\rangle = 0$; or, it is an eigenstate of a with 0 eigenvalue.

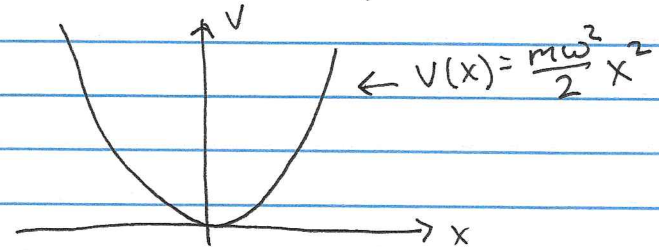
The complete spectrum of energy eigenstates can be constructed by acting with $a^\dagger$ on $|\psi_0\rangle$. We had found the n^{th} eigenstate to be:

$$|\psi_n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|\psi_0\rangle \quad \text{where } \hat{H}|\psi_n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega.$$

This state is normalized: $\langle\psi_n|\psi_n\rangle = 1$.

Today, we're going to revisit an especially odd feature of the harmonic oscillator, but was also something we observed in the infinite square well. Apparently, the ground state, the lowest possible energy state, has <u>non-zero</u> energy. Why? Why is 0 energy not possible <u>quantum mechanically</u> while it is classically?

Let's consider what $E=0$ would mean. First, the harmonic oscillator potential looks like:



There is only <u>one</u> point on the potential where its energy is 0: at $x=0$. All other points have greater energy, so if we say that the total energy is 0, we must require $x=0$. Further, the kinetic energy is a square of momentum, so it is necessarily non-negative: $K = \frac{p^2}{2m} \geq 0$. However, for $K=0$, we must enforce that $p=0$ as well. Thus, the <u>only</u> possible way for $E=0$ is if both $x=0$ <u>and</u> $p=0$.

If this is true, then note that such a state has 0 variance in both x and p: $\sigma_x = \sigma_p = 0$. However, now I think you see the problem: such a state violates the <u>very</u> general bound we had established, the Heisenberg uncertainty relation. Because position and momentum do <u>not</u> commute, it is not possible for a state to have both $x=0$ and $p=0$, and thus a 0 energy state of the harmonic oscillator <u>can not</u> exist.

Okay, but what is the ground state doing? Why is it is what it is? Let's calculate the momentum and position variances, σ_p^2 and σ_x^2, and see what its corresponding product is. The first thing we need to do is to express operators $\hat{x}$ and $\hat{p}$ in terms of a and a^+. From the relationships earlier, we find:

$$\hat{x} = \sqrt{\frac{2\hbar}{m\omega}}\left(\frac{a+a^+}{2}\right), \quad \hat{p} = i\sqrt{2m\hbar\omega}\left(\frac{a^+-a}{2}\right) \quad \text{or}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a+a^+\right), \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^+-a\right)$$

On the ground state, $|\psi_0\rangle$, let's calculate σ_x^2, σ_p^2. Starting with σ_x^2, we have:

$$\sigma_x^2 = \langle\psi_0|\hat{x}^2|\psi_0\rangle - \langle\psi_0|\hat{x}|\psi_0\rangle^2$$

Note that the expectation value of $\hat{x}$ is 0:

$$\langle\psi_0|\hat{x}|\psi_0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\psi_0|a+a^+|\psi_0\rangle = 0,$$

because $a|\psi_0\rangle = 0$ and $\left(a|\psi_0\rangle\right)^+ = \langle\psi_0|a^+ = 0$

This can be used to simplify the expectation value of $\hat{x}^2$:

$$\langle\psi_0|\hat{x}^2|\psi_0\rangle = \langle\psi_0|\frac{\hbar}{2m\omega}\left(a+a^+\right)^2|\psi_0\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_0|\cancelto{0}{a^2} + \cancelto{0}{(a^+)^2} + aa^+ + \cancelto{0}{a^+a}|\psi_0\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_0|aa^+|\psi_0\rangle = \frac{\hbar}{2m\omega}\langle\psi_0|\cancelto{0}{a^+a} + \cancelto{1}{[a,a^+]}|\psi_0\rangle$$

$$= \frac{\hbar}{2m\omega}, \text{ because } |\psi_0\rangle \text{ is normalized: } \langle\psi_0|\psi_0\rangle = 1.$$

Now, let's do the same thing for momentum, $\hat{p}$. Again, the expectation value of $\hat{p}$ vanishes, for the same reason as $\hat{x}$. The expectation value of $\hat{p}^2$ is:

$$\langle\psi_0|\hat{p}^2|\psi_0\rangle = -\langle\psi_0|\frac{m\hbar\omega}{2}(a^\dagger - a)^2|\psi_0\rangle = -\frac{m\hbar\omega}{2}\langle\psi_0|\cancelto{0}{(a^\dagger)^2} + \cancelto{0}{(a)^2} - \cancelto{0}{a^\dagger a} - aa^\dagger|\psi_0\rangle$$

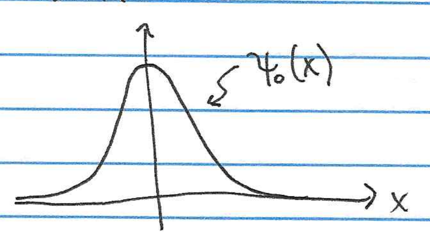
$$= \frac{m\hbar\omega}{2}\langle\psi_0|aa^\dagger|\psi_0\rangle = \frac{m\hbar\omega}{2}.$$

So, we find that the product of variances of $\hat{x}$ and $\hat{p}$ are:

$$\sigma_p^2\sigma_x^2 = \langle\psi_0|\hat{p}^2|\psi_0\rangle\langle\psi_0|\hat{x}^2|\psi_0\rangle = \frac{m\hbar\omega}{2}\cdot\frac{\hbar}{2m\omega} = \frac{\hbar^2}{4}.$$

Thus, the ground state of the harmonic oscillator saturates the Heisenberg uncertainty principle! Thus, one interpretation of why the ground state energy is non-zero, is to satisfy Heisenberg uncertainty. If the ground state energy were lower, then the variance on momentum and position must decrease, but this is forbidden by the commutation relation of $\hat{x}$ and $\hat{p}$.

Let's understand this minimum uncertainty state a bit more. Recall that, in position space, the ground state wavefunction was a Gaussian:



The variance of this Gaussian is apparently $\sigma_x = \frac{\hbar}{2m\omega}$. We can construct other minimal uncertainty states by noting the following property of the variance.

The variance of some state is just the measure of the distribution about the mean of the distribution. The variance doesn't care about the specific value of the mean; in the case of the Gaussian, the variance doesn't care where the peak is. Another way to say this is that the variance is <u>translationally-invariant</u>. From the definition of the variance:

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2,$$

we see this translational invariance by replacing $x \to x-\lambda$, for some number λ. We see the variance transforms to:

$$\begin{aligned}\sigma_x^2 &\to \langle (x-\lambda)^2\rangle - \langle x-\lambda\rangle^2 \\ &= \langle x^2 - 2\lambda x + \lambda^2\rangle - \langle x\rangle^2 + 2\lambda\langle x\rangle - \lambda^2 \\ &= \langle x^2\rangle - \cancel{2\lambda\langle x\rangle} + \cancel{\lambda^2} - \langle x\rangle^2 + \cancel{2\lambda\langle x\rangle} - \cancel{\lambda^2} \\ &= \langle x^2\rangle - \langle x\rangle^2\end{aligned}$$

That is, the ground state Gaussian and the Gaussian centered at $x=\lambda$ have the <u>same</u> variance, and thus have the same uncertainty relation.

Using the $a, a^\dagger$ expression for $\hat{x}$, we note that translation of $\hat{x}$: $\hat{x} \to \hat{x} - \lambda$, is equivalent to just translating a:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) \to \sqrt{\frac{\hbar}{2m\omega}}(a-\lambda+a^\dagger) = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) - \lambda\sqrt{\frac{\hbar}{2m\omega}}.$$

Further, the ground state $|\psi_0\rangle$ was an eigenstate of a with 0 eigenvalue: $a|\psi_0\rangle = 0|\psi_0\rangle$, so if we consider a state $|\chi\rangle$ which is annihilated by the shifted a:

$$(a-\lambda)|\chi\rangle = 0 \Rightarrow a|\chi\rangle = \lambda|\chi\rangle,$$

this is just an eigenstate of a with some eigenvalue λ. Recall that a is <u>not</u> Hermitian, so λ is in general a complex number. Does such an eigenstate of a correspond to a minimum uncertain state? Let's just calculate and see.

First, we calculate the expectation value of $\hat{x}$:

$$\langle \chi | \hat{x} | \chi \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \chi | a + a^\dagger | \chi \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\lambda + \lambda^*)$$

Now, the square of $\hat{x}$:

$$\langle \chi | \hat{x}^2 | \chi \rangle = \frac{\hbar}{2m\omega} \langle \chi | a^2 + (a^\dagger)^2 + a a^\dagger + a^\dagger a | \chi \rangle$$

$$= \frac{\hbar}{2m\omega} \left(\lambda^2 + (\lambda^*)^2 + \lambda^* \lambda + \langle \chi | a a^\dagger | \chi \rangle \right)$$

$$= \frac{\hbar}{2m\omega} \left(\lambda^2 + (\lambda^*)^2 + \lambda^* \lambda + \langle \chi | a^\dagger a + [a, a^\dagger] | \chi \rangle \right)$$

$$= \frac{\hbar}{2m\omega} \left((\lambda + \lambda^*)^2 + 1 \right)$$

So, the variance of x is:

$$\sigma_x^2 = \langle \chi | \hat{x}^2 | \chi \rangle - \langle \chi | \hat{x} | \chi \rangle^2 = \frac{\hbar}{2m\omega} \left[(\lambda + \lambda^*)^2 + 1 - (\lambda + \lambda^*)^2 \right]$$

$= \frac{\hbar}{2m\omega}$, which is the same as the ground state, $|\psi_0\rangle$!

While I won't do it here, you can verify the same thing holds for the variance of p, $\sigma_{\hat{p}}^2$. Thus, we find that eigenstates of the lowering/annihilation operator a are minimal uncertainty states. Further, because the Heisenberg uncertainty principle is exclusively a quantum phenomena, we say that states that saturate the bound are "maximally quantum"!

As a final thing for this lecture, I want to explicitly construct this state $|\chi\rangle$. To do this, we can simply write it as some linear combination of the energy eigenstates

$|\psi_n\rangle$, assuming completeness. Then, we can write

$$|\chi\rangle = \sum_{n=0}^{\infty} \beta_n |\psi_n\rangle = \sum_{n=0}^{\infty} \beta_n \frac{(a^+)^n}{\sqrt{n!}} |\psi_0\rangle,$$

where β_n is some complex number and in the second equality we used the explicit form of $|\psi_n\rangle$ as a^+ acting on the ground state $|\psi_0\rangle$. To determine β_n, we just require that this is an eigenstate of a. Acting on $|\chi\rangle$ with a, we have:

$$a|\chi\rangle = \sum_{n=0}^{\infty} \frac{\beta_n}{\sqrt{n!}} a(a^+)^n |\psi_0\rangle = \sum_{n=0}^{\infty} \frac{\beta_n}{\sqrt{n!}} \Big((a^+)^n a + [a, (a^+)^n] \Big) |\psi_0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{\beta_n}{\sqrt{n!}} n (a^+)^{n-1} |\psi_0\rangle = \sum_{n=0}^{\infty} \lambda \beta_n \frac{(a^+)^n}{\sqrt{n!}} |\psi_0\rangle.$$

In evaluating this, we used the commutation relations we derived earlier this week. The final equality is simply the eigenvalue λ times $|\chi\rangle$.

We can make the comparison more direct ~~a~~ as writing the action of a on $|\chi\rangle$ indexed from the ground state as:

$$\sum_{n=0}^{\infty} \frac{\beta_n}{\sqrt{n!}} n (a^+)^{n-1} |\psi_0\rangle = \sum_{n=0}^{\infty} \frac{\beta_{n+1}}{\sqrt{(n+1)!}} (n+1) (a^+)^n |\psi_0\rangle$$

Thus, comparing to the expression of $\lambda|\chi\rangle$, we find the recursion relation for the β_n:

$$\frac{\lambda \beta_n}{\sqrt{n!}} = \frac{\beta_{n+1} \sqrt{n+1}}{\sqrt{n!}}, \text{ or that } \beta_{n+1} = \frac{\lambda \beta_n}{\sqrt{n+1}}.$$

This is solved by: $\beta_n = \frac{\lambda^n}{\sqrt{n!}} \beta_0$, for some initial coefficient β_0.

Then, this eigenstate of a can be expressed as:

$$|\chi\rangle = \sum_{n=0}^{\infty} \beta_0 \frac{\lambda^n}{n!} (a^+)^n |\psi_0\rangle = \left(\sum_{n=0}^{\infty} \frac{(\lambda a^+)^n}{n!} \right) \beta_0 |\psi_0\rangle$$

$$= e^{\lambda a^+} \beta_0 |\psi_0\rangle.$$

Apparently these maximally quantum states can be created by acting this exponentiated a^+ operator. Such states are called "coherent states", and repesent a Gaussian ($|\psi_0\rangle$) displaced from $x=0$ and $p=0$ by an amount determined by λ. The number β_0 can be found by enforcing normalization of $|\chi\rangle$. Coherent states appear all over in physics: from optics to particle physics, and they have interesting properties that make them ideal for the model quantum state. You'll explore some of these properties in homework.

Have a good weekend!