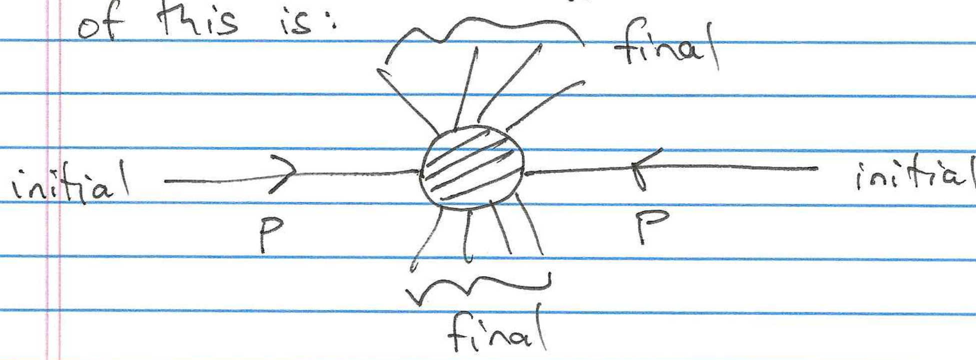
Physics 342 Lecture 21

Welcome to Friday of week 7! Please turn in homework.

Last lecture, we had introduced the phenomena of scattering: Incident momentum eigenstates on a localized potential that then scatter, either transmitting or reflecting. One can argue, and I will do so in this lecture, that scattering is the most fundamental way in which we interact with the world in general. Let's focus on vision, for example. Light from the sun travels essentially unimpeded (ie., free) until it hits something localized, like a blade of grass. Some amount of that light is transmitted or absorbed by the grass, and the rest is reflected into your eye, say. Your brain performs complicated analysis of this reflected light to interpret it as, indeed, a blade of grass. That is, we don't <u>see</u> the grass: we see and reinterpret the light reflected from the grass.

This idea is also explored in particle physics experiments, like the Large Hadron Collider at CERN in Geneva, Switzerland. The goal of the large Hadron collider, or LHC, is to determine how particles interact at length scales of about 10^{-20} m. This is waaaaaay too small to look with light: visible light has a wavelength of hundreds of nanometers, about a trillion times larger than the distances the LHC probes. Instead, the LHC accelerates protons to very high energies and collides them head on. Then, after collision, numerous particles are produced, according to the interactions that happened at the

10^{-20} m distance scale. Enormous particle detectors are built around these collisions or interaction points to measure all (or nearly all) of the particles produced. Then, the goal of the scientist is the inverse problem: given the initial proton momenta and the momenta of all of the final particles, what interactions happened in between? The picture of this is:



The blob in the middle represents the interactions responsible for turning the protons into whatever the final particles are.

While we don't immediately know what the blob is, quantum mechanics already tells us a lot about it. First, we can represent the initial state of the two protons as a Dirac ket, called an in state:

$$|\text{protons}\rangle \equiv |\text{in}\rangle$$

As a quantum state in the Hilbert space, it must be normalized:

$$\langle \text{in}|\text{in}\rangle = 1.$$

Further, the collection of final state particles can also be represented as a ket, called an out state:

$$|\text{final particles}\rangle \equiv |\text{out}\rangle.$$

Again, as a state in the Hilbert space, it must be normalized:

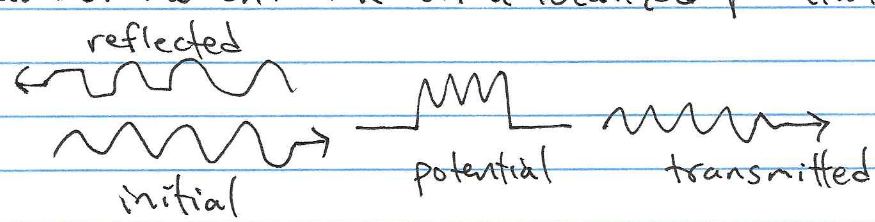
$$\langle out | out \rangle = 1.$$

These in and out states are both states in the same Hilbert space, as they happen in the same universe. Therefore, there is some unitary operator S that transforms the in state to an out state:

$$S|in\rangle = |out\rangle.$$

Conservation of probability requires that $S^\dagger S = \mathbb{1}$, the identity matrix. The unitary matrix S is called the S-matrix, short for scattering matrix. Note that the elements of the scattering matrix are observable, measurable: I know the in state (I prepare it) and I measure the out state, so I can infer S. As a particle physicist myself, my goal is to take the S-matrix and use it to determine it to figure out the physics located at 10^{-20} m.

Now, in this class we won't discuss the S-matrix relevant for the LHC, but we can for scattering off of a localized, quantum mechanical potential as introduced last lecture. Last time, if you recall, we considered the scattering of an initial right-moving wave of momentum k on a localized potential:



We had calculated the amplitudes of the reflected and transmitted waves and from them constructed transmission and reflection coefficients. In principle, the dependence of T and R on the initial wave's momentum k can be used to determine what the potential was. For the case of a constant potential:

$$V(x) = \begin{cases} V_0, & 0 < x < a \\ 0, & \text{else} \end{cases}$$

we had found the reflection and transmission amplitudes A_R and A_T to be:

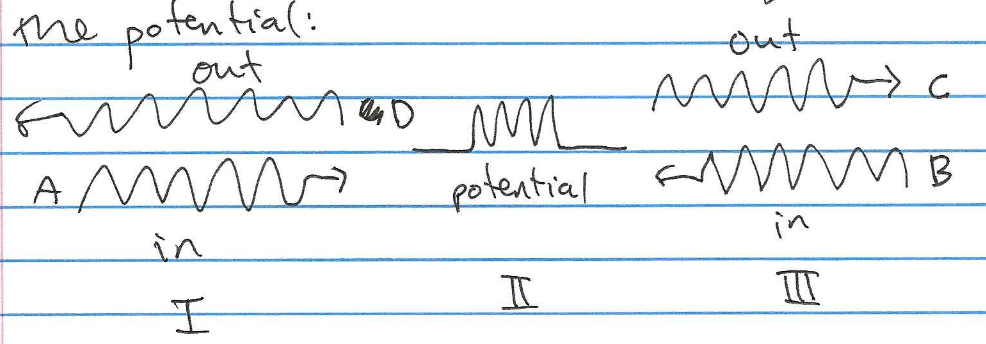
$$A_R = \frac{mV_0}{k^2 - mV_0 + ik\sqrt{k^2-2mV_0}\,\cot\left(\frac{a}{\hbar}\sqrt{k^2-2mV_0}\right)}$$

$$A_T = \frac{k\sqrt{k^2-2mV_0}}{e^{i\frac{ak}{\hbar}}\,k\sqrt{k^2-2mV_0}\,\cos\left(\frac{a}{\hbar}\sqrt{k^2-2mV_0}\right) - ie^{i\frac{ak}{\hbar}}\left(k^2-mV_0\right)\sin\left(\frac{a}{\hbar}\sqrt{k^2-2mV_0}\right)}$$

One can verify that these conserve probability; that is,

$$T + R = |A_T|^2 + |A_R|^2 = 1.$$

Now we want to consider the more general case in which waves from both the left and right are incident on the potential:



Just as with one wave scattering, we determine the wavefunction in each region and match across boundaries.

In Region I, we have: $\psi_I(x) = Ae^{\frac{ikx}{\hbar}} + De^{-\frac{ikx}{\hbar}}$,

while in Region III we have: $\psi_{III}(x) = Be^{-\frac{ikx}{\hbar}} + Ce^{\frac{ikx}{\hbar}}$

Note that A and B represent the "in" waves while C and D represent the "out" waves. By probability conservation, there must be a unitary matrix S that connects A, B to C, D. This is just the S-matrix for scattering off of a potential:

$$\begin{pmatrix} C \\ D \end{pmatrix} = S\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} S_{AC} & S_{BC} \\ S_{AD} & S_{BD} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \Leftrightarrow S|in\rangle = |out\rangle,$$

where we have written out the individual matrix elements, with subscripts that represent the amplitudes for the respective in wave to reflect or transmit the out wave. In this representation, transmission is represented by the diagonal elements S_{AC}, S_{BD}, and reflection is the off diagonal elements. The S-matrix is unitary, which enforces relationships between elements:

$$S^\dagger S = \mathbb{1} \Rightarrow \begin{pmatrix} S_{AC}^* & S_{AD}^* \\ S_{BC}^* & S_{BD}^* \end{pmatrix}\begin{pmatrix} S_{AC} & S_{BC} \\ S_{AD} & S_{BD} \end{pmatrix} = \begin{pmatrix} |S_{AC}|^2 + |S_{AD}|^2 & S_{AC}^* S_{BC} + S_{AD}^* S_{BD} \\ S_{AC} S_{BC}^* + S_{AD} S_{BD}^* & |S_{BC}|^2 + |S_{BD}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we again take the constant potential to scatter off, it is straightforward to determine the S-matrix elements. We won't go through the details here, but what we do is ensure that the wavefunction is continuous and smooth across each boundary: I to II and II to III.

For the constant potential, we can express the wavefunction in region II as:

$$\psi_{II}(x) = F e^{\frac{i}{\hbar}\sqrt{2m(E-V_0)}\,x} + G e^{-\frac{i}{\hbar}\sqrt{2m(E-V_0)}\,x},$$

where $E = \frac{k^2}{2m}$ and F and G are some complex coefficients. Note that we already know two of the S-matrix elements, because we have studied the case when $A=1$, $B=0$:

$S_{AC} = A_T$, $S_{AD} = A_R$, from earlier. Unitarity

requires that: $|S_{BD}| = |S_{AC}| = |A_T|$ and $|S_{BC}| = |S_{AD}| = |A_R|$

and $S_{AC}^* S_{BC} + S_{AD}^* S_{BD} = A_T^* S_{BC} + A_R^* S_{BD} = 0$. So, all we do not

know about S_{BC} and S_{BD} are their phases. That is, we can write:

$$S_{BC} = e^{i\phi_1}|A_R| \text{ and } S_{BD} = e^{i\phi_2}|A_T|,$$

for some real numbers ϕ_1, ϕ_2. Again, these can be calculated using the matching conditions, but we won't do that here. For our purposes it will be enough to note that the S-matrix can be written as:

$$S = \begin{pmatrix} A_T & e^{i\phi_1}|A_R| \\ A_R & e^{i\phi_2}|A_T| \end{pmatrix}$$

In the rest of this lecture we will note a couple of features of this S-matrix as an example of more general phenomena.

The first thing to note is that there are special values of k for which $A_R = 0$; that is, the potential is <u>transparent</u>. In the denominator of A_R, we had a cotangent factor that looked like

$$\cot\left(\frac{a}{\hbar}\sqrt{k^2-2mV_0}\right)$$

Cotangent diverges if its argument is an integer multiple of π, and if it diverges in the denominator, then A_R vanishes. That is, the potential is transparent if:

$$n\pi = \frac{a}{\hbar}\sqrt{k^2-2mV_0}\,, \text{ or if } k = \sqrt{\frac{n^2\pi^2\hbar^2}{a^2}+2mV_0}$$

Let's massage this a bit more. Note that the momentum when the particle is over the potential is $\sqrt{k^2-2mV_0}$, so the energy is

$$E = \frac{k^2 \cancel{-2mV_0}}{2m}\,,$$ or, in terms of n,

$$E = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{k^2 \cancel{-2mV_0}}{2m}.$$ Have we seen this energy before? Yes! The ~~bound~~ energy levels of the infinite square well! Note that if $V_0 < 0$, that is, the potential is a <u>finite square well</u>. So this is suggestive that the potential becomes transparent when the momentum k is in phase or <u>resonant</u> with a bound state of the finite square well. You'll explore this more in homework, but in general, the potential is transparent when you are in resonance.

Additionally, we can rewrite the S-matrix in a useful form. Let's write it as:

$$S = \mathbb{1} + i\mathcal{M},$$ where $\mathbb{1}$ is the

identity matrix and $\mathcal{M}$ is another matrix. The identity matrix component of the S matrix means that no scattering took place: the potential was transparent and nothing was reflected. Thus, the matrix $\mathcal{M}$ represents the interesting scattering content of the S-matrix, and the part that encodes all information about the potential.

It's interesting to consider the unitarity of S written in this way. Note that $S^\dagger$ is:

$$S^\dagger = \mathbb{1} - i\mathcal{M}^\dagger, \text{ so that}$$

$$S^\dagger S = \mathbb{1} = (\mathbb{1} - i\mathcal{M}^\dagger)(\mathbb{1} + i\mathcal{M}) = \mathbb{1} + i(\mathcal{M} - \mathcal{M}^\dagger) + \mathcal{M}^\dagger\mathcal{M}.$$

For S to be unitary, we must enforce that $\mathcal{M}$ satisfies:

$$\mathcal{M}^\dagger\mathcal{M} = -i(\mathcal{M} - \mathcal{M}^\dagger) = 2\frac{(\mathcal{M} - \mathcal{M}^\dagger)}{2i} = 2\,\mathrm{Im}(\mathcal{M}).$$

This result, that $\mathcal{M}^\dagger\mathcal{M} = 2\,\mathrm{Im}(\mathcal{M})$ is called the <u>optical theorem</u>: the probability for non-trivial scattering $(\mathcal{M}^\dagger\mathcal{M})$ is equal to the imaginary part of the amplitude for that same non-trivial scattering. This highly non-trivial relationship between probabilities and amplitudes will be studied more in homework.

That's it for scattering in this class; next week, we go up to three dimensions and start talking about angular momentum...