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Physics 342 Lecture 22

Welcome back! I hope you had a good weekend. Only one more week before spring break, and I know I need a break to refresh!

With scattering and its formalism from last week, we are essentially done with studying quantum mechanics in one dimension. For the rest of this course, we will work to generalize our formulation of quantum mechanics to be more realistic; i.e., actually account for the multiple spatial dimensions that we experience in our universe. This week, we will introduce a profound consequence of living in multiple spatial dimensions and its consequences for a study of angular momentum in quantum mechanics. In this lecture, we will mostly set the stage for describing rotations and later this week, construct the complete theory of angular momentum; at least as much as we need here.

To start, let's quantify the transformations of quantum mechanics in one dimension and see what that can get us. First, we restrict ourselves to linear operations/transformations as this is quantum mechanics, and further, transformations that keep us in the Hilbert space, i.e., unitary operations. In one spatial and one time dimension, there are only three linear transformations that one can possibly consider: We've seen two of them: we can translate in space. This is accomplished by the unitary operator:

$$\mathbb{U}_p(x) = e^{\frac{i\hat{p}x}{\hbar}},$$

which translates by a distance x, according to the

Hermitian momentum operator $\hat{p}$. We can also translate in time with the unitary operator:

$$U_{\hat{H}}(t) = e^{-\frac{i\hat{H}t}{\hbar}},$$ where t is the time translation,

and $\hat{H}$ is the Hermitian Hamiltonian operator. Now, I said there were three things you can do to space and time, but these were only two transformations we discussed. Another thing that is 1) linear and 2) can be implemented by a unitary operator is a transformation that produces a linear combination of space and time. For concreteness, let's construct a two-component vector with first entry time t and second entry position x:

$$\vec{y} = \begin{pmatrix} t \\ x \end{pmatrix}$$

Now, we can mix space and time with a matrix such as:

$$U_\eta = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}$$

Now, acting U_η on $\vec{y}$ produces a Lorentz transformation or Lorentz boost and η is called the ~~pa~~ rapidity. Now, if you're paying attention at home you might realize that this is not a unitary matrix, and its non-unitarity (or apparent non-unitarity) comes from the fact that there is a well-known "$-$" sign in special relativity between space and time. Relativistic quantum mechanics is not a topic of this course, but I wanted you to be aware of the starting point. Just like non-relativistic quantum mechanics that we are studying here starts with identification of the relevant unitary/Hermitian operators that enact transformations ~~on~~ of states in our Hilbert space, the same thing is the starting point for a relativistic theory.

So, with no mixing of space and time allowed in this class, these translations of space and time individually is all we can do in 1D. Now, if we even consider two spatial dimensions, things get much more interesting. First, in two spatial dimensions x and y, say, we can translate in either dimension, so we now have two components of a Hermitian momentum vector $\hat{p} = (\hat{p}_x, \hat{p}_y)$ where

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}.$$

Because partial derivatives commute, note that $\hat{p}_x$ and $\hat{p}_y$ commute with each other: $[\hat{p}_x, \hat{p}_y] = 0$.

However, this isn't everything. Just like we could mix space and time, we can in principle mix the x dimension and the y dimension with some linear combination. If additionally we want that linear combination to be unitary and keep positions real valued, we are uniquely led to the possible transformation. If we call the vector $\vec{r}$ the position vector, where

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix},$$

then the only possible real, unitary transformation is:

$$\mathbb{U}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some angle θ. As we have studied this is indeed unitary.

As a unitary operator, it of course can be written as the exponentiation of some Hermitian operator we will call $\hat{L}$. What is this $\hat{L}$?

Well, we want to write $\mathbb{U}_\theta = e^{+i\theta\hat{L}} = \mathbb{1} + i\theta\hat{L} + \dots$ where we have Taylor expanded in θ on the right. To determine $\hat{L}$, we can just Taylor expand the action of $\mathbb{U}_\theta$ on the vector $\vec{r}$:

$$\mathbb{U}_\theta \vec{r} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} x - \theta y + \dots \\ y + \theta x + \dots \end{pmatrix} = r + \theta \begin{pmatrix} -y \\ x \end{pmatrix} + \dots$$

$$= \left(\mathbb{1} + i\theta\hat{L}\right)\vec{r} + \dots .$$

So, equating these expressions, we apparently must have

$$i\theta\hat{L}\vec{r} = i\theta\hat{L}\begin{pmatrix} x \\ y \end{pmatrix} = \theta\begin{pmatrix} -y \\ x \end{pmatrix}$$, or by canceling θ and multiplying by $-i$:

$$\hat{L}\begin{pmatrix} x \\ y \end{pmatrix} = -i\begin{pmatrix} -y \\ x \end{pmatrix}.$$ Now this may look strange, but you can easily write down a 2x2 matrix that accomplishes this.

However, I want to do something different. Let's focus on the first entry:

$\hat{L}x = iy$. What linear operations allow us to turn x into y? Well, differentiating wrt x returns 1:

$\frac{\partial}{\partial x} x = 1$, so we just multiply by iy:

$\hat{L}x = iy\frac{\partial}{\partial x}x = iy$. Similarly, we can turn y into $-ix$:

$\hat{L}y = -ix\frac{\partial}{\partial y}y = -ix$. Now, as $\frac{\partial}{\partial x}y = \frac{\partial}{\partial y}x = 0$, we can

combine these into a single operator:

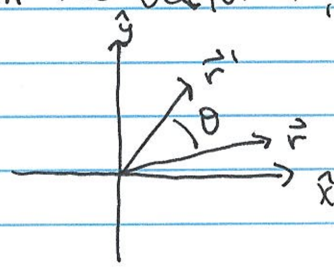
$$\hat{L} = -ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$

Where we have re-expressed the operator $\hat{L}$ in terms of momentum and position operators. Now, even from freshman physics this should look very familiar: this is just angular momentum expressed like an operator cross product. We say in quantum mechanics that angular momentum $\hat{L}$ generates rotations in space. Note also that $\hat{L}$ is indeed Hermitian because $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ are all Hermitian.

We'll come back to angular momentum and get to three dimensions next lecture, but for the rest of this lecture I just want to focus on this unitary rotation matrix, $\mathbb{U}(\theta)$:

$$\mathbb{U}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Acting on the vector $\vec{r}$, this of course rotates $\vec{r}$ by angle θ:



where $\vec{r}\,' = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$

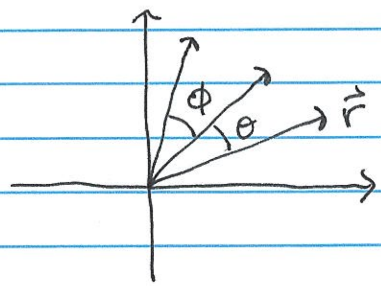
Now, what if we further rotate by ϕ with the matrix $\mathbb{U}(\phi)$? Then, we have:

$$\mathbb{U}(\phi)\,\mathbb{U}(\theta)\,\vec{r} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In the final line I have used angle addition formulas. (Actually, this is a way to remember them!) Not surprisingly, rotating a two-dimensional vector an angle θ then ϕ is equivalent to just rotation by $\theta+\phi$:



Further, it didn't matter what order we performed the rotations, ϕ first or θ first:

$$\mathbb{U}(\theta)\mathbb{U}(\phi)=\mathbb{U}(\phi)\mathbb{U}(\theta)=\mathbb{U}(\theta+\phi)=\mathbb{U}(\phi+\theta)$$

That is we say that rotations of two-dimensional vectors are commutative. We have also identified the multiplication law of rotations, as expressed above. This rich structure of rotations actually forms a mathematical object called a group. A group is a set of objects that has a multiplication operator $\cdot$ and that set and multiplication satisfy the following four properties:

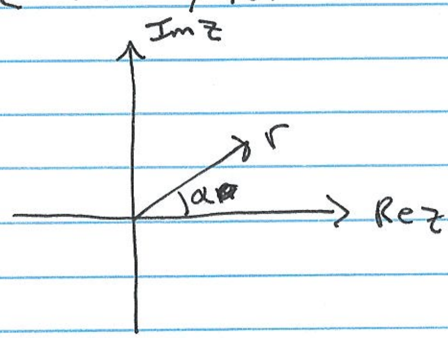
1) Closure. If a & b are in the group, $a\cdot b$ is in the group.
2) Identity. $\exists$ an element of the group 1 such that $1\cdot a = a\cdot 1 = a$ for another group element a.
3) Inverse. $\forall$ element a, $\exists$ an element a^{-1} for which $a\cdot a^{-1} = a^{-1}\cdot a = 1$
4) Associtivity. for three group elements a, b, c, multiplication is associative: $a\cdot(b\cdot c) = (a\cdot b)\cdot c$.

I claim that these two-dimensional rotations form a group, and you'll study that more in homework. We call this 2D rotation group Abelian, because all elements commute via matrix multiplication. Commutativity is not a requirement to be a group, and we'll soon see examples of when that is not the case.

To end today, I want to emphasize something. If you know the multiplication law for a group, you know everything there is to know about the group. This feature of groups enables many different representations of how that multiplication law is manifest. We saw how rotations acted on the two-dimensional vector $\vec{r}$, but we can represent a point in the x-y plane in many ways. For example, we could say that x and y are the real and imaginary parts of a complex number z:

$z = x+iy$. We can equivalently write this as

$z = re^{i\alpha}$, for some length $r=\sqrt{x^2+y^2}$ and phase α:



How do we rotate this representation of the vector? Well all we do is multiply by an appropriate exponential phase! to rotate by θ, we just do:

$$re^{i\alpha} \rightarrow re^{i\alpha}e^{i\theta} = re^{i(\alpha+\theta)}$$

Rotating again by ϕ is just as trivial:

$$re^{i\alpha} \rightarrow re^{i\alpha}e^{i\theta} \rightarrow re^{i\alpha}e^{i\theta}e^{i\phi} = re^{i\alpha}e^{i(\theta+\phi)}.$$

Let's call $A(\theta) = e^{i\theta}$ so that we have shown

$$A(\theta)A(\phi) = A(\phi)A(\theta) = A(\theta+\phi) = A(\phi+\theta).$$

But this is the <u>identical</u> multiplication law that we found for the 2x2 rotation matrix! Thus, as groups, we say that the action of multiplication by an exponential phase is equivalent to that of 2x2 rotation matrices. The former group is called $U(1)$ (1x1 unitary matrices) and the latter group is called $SO(2)$ (special orthogonal 2x2 matrices) so we found that:

$$U(1) \simeq SO(2)$$, where "$\simeq$" means equivalent as groups.

We have thus found two different <u>representations</u> of the same group.

We'll see this group theory formulation and representations of groups front and center when we consider rotations in three dimensions.