Physics 342 Lecture 23

Welcome back for more quantum mechanics! In the previous lecture, we had studied rotations of the two-dimensional plane. For the two-dimensional position vector $\vec{r}$, we had identified the rotation matrix $\mathbb{U}(\theta)$ as that that transforms $\vec{r}$:

$$\vec{r}\,' = \mathbb{U}(\theta)\,\vec{r}\,, \quad \text{where} \quad \mathbb{U}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This matrix is unitary which is a requirement to be a map of the Hilbert space to itself. As a unitary matrix, it can be represented as the exponential of a hermitian operator, $\hat{L}$, and we had found that:

$$\mathbb{U}(\theta) = e^{i\theta\hat{L}}\,, \quad \text{where } \hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$ the effective "cross product" of the position and momentum operators.

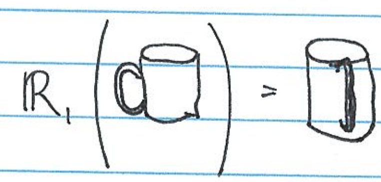
This rich structure already suggested that there was deeper mathematics lurking below the surface. We got our first glimpse of <u>groups</u> and <u>representations</u> in understanding the properties of composition of subsequent rotations. Rotations in two-dimensions formed a group, and further one that is Abelian, or commutative, where the multiplication order of group elements is irrelevant:

$$\mathbb{U}(\theta)\,\mathbb{U}(\phi) = \mathbb{U}(\phi)\,\mathbb{U}(\theta) = \mathbb{U}(\theta+\phi)$$

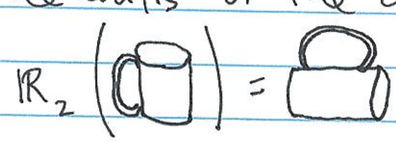
Two-dimensional rotations only get us so far because we of course, live in three-dimensions, and not two. So, we'll start this lecture with an enumeration of some interesting properties of rotations in three-dimensions.

One thing we noted when introducing groups on Monday was that the property of commutivity was <u>not</u> implied by the four group properties. We happened to see that two-dimension rotations <u>do</u> commute, just like multiplication of familiar numbers. However, just like two generic matrices $\mathbb{A}$ and $\mathbb{B}$ do not commute: $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$, we shouldn't expect that the subsequent action of rotations in three dimensions commutes. So, to make this concrete, let's consider a ~~thee~~ three-dimensional object and just see how it rotates. I have here my coffee cup, which is a convenient prop because the handle serves as a nice reference for the orientation of the cup. Now, we can rotate this thing around however we want, like so, and you are able to put it in whatever orientation you want by appropriate rotation.

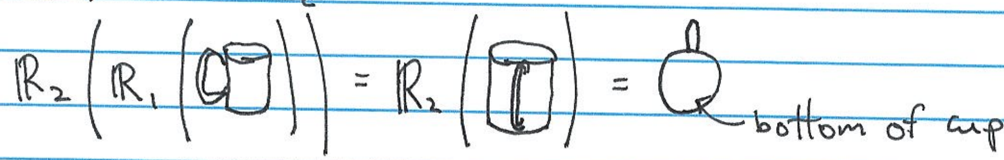
What we will do now is consider what happens when we perform two ~~su~~ subsequent rotations in different orders. One rotation, call it $\mathbb{R}_1$, rotates the cup 90° about the axis that passes through the center of the cup as thought of as a cylinder:



The other rotation we will perform, call it $\mathbb{R}_2$, is a 90° rotation of the cup about an axis that passes through the "walls" of the cup:



Okay, let's see what happens if we perform $\mathbb{R}_1$ first, then $\mathbb{R}_2$:



What about $\mathbb{R}_2$ first <u>then</u> $\mathbb{R}_1$? (arr-too!)

Are the implementations of these rotations in different orders the same? No! Apparently three dimensional rotations, unlike rotations in two-dimensions, are not commutative, or <u>non-Abelian</u>:

$$\mathbb{R}_2 \mathbb{R}_1 \neq \mathbb{R}_1 \mathbb{R}_2 .$$

Again, we didn't require it to be Abelian and didn't necessarily expect it to be, but it is perhaps somewhat shocking to see it so explicitly.

The next property of three-dimensional rotations I hope is <u>extremely</u> surprising. To set the stage, let's go back to just two dimensional rotations. As you well know, if you rotate by 2π, you get back to where you started, as exhibited by me rotating in place here. But wait: you do get back where you start, in the sense of every 2π rotation is periodic, but this is very different than <u>equating</u> a 2π rotation with no rotation at all. Indeed, if I rotate in place by 2π, my angle is 2π; if I go another 2π, my angle is 4π, etc. By continuing to rotate in the same direction I can never get back to no rotation; i.e., 0 angle. A concrete way to track this is to wind a string around a cylinder/cardboard roll. If you wind the string by 2π, then indeed a point on the string gets back to where you started, but you can never unwind the string by wrapping in the same direction:

This is a concrete manifestation of the property that the group of two-dimensional rotations is <u>not</u> <u>simply-connected</u>: there exist orbits in group space (i.e., trajectories from repeated action of group elements) that, while they return the object to the original orientation, they are <u>not</u> equivalent to the identity, no rotation at all. This is not so surprising: we had identified the group of two dimensional rotations as $SO(2) \simeq U(1)$, and $U(1)$ can be identified with the unit circle in the complex plane:

$$e^{i\phi} \in U(1), \quad \phi \in \mathbb{R}.$$

A circle clearly has a hole in it, so another colloquial way to thing about ~~simp~~ connectedness and simple vs. not simple is (simply): if the group/set of interest has a "hole" then it is not simply-connected. Simply connected sets have no holes. This is also why the cylinder works to track how rotations accumulate as you wind the string.

So, this was a long winded way around. What about three-dimensional rotations? Well, you might think that essentially the same argument would hold for three-dimensions and therefore three-dimensional rotations would not be simply connected either. However, in three-dimensions one has "above-ness" and "below-ness" as an additional orientation between objects, and this will make all the difference.

To test the connectedness of the set of all three-dimensional rotations, there's a profound demonstration that you can do with your body, rather than a cylinder and a string. We'll use your arm like the string to keep track of the orbit that you perform as you rotate. So, here's what you do. Put your arm out, palm up, like you are balancing a plate on it. In what follows, always keep your palm up: you don't want to drop the plate! Now, rotate your hand about a vertical axis that passes through your hand. Pass your hand below your arm, and rotate it through 2π. If you did it right, your palm should be in the original orientation, but your arm is all twisted! So, apparently, just like two-dimensional rotations, in 3D, $2\pi \neq 0$.

However, we aren't done. With your hand in this twisted orientation, continue to rotate your hand in the same direction as earlier, but now pass your hand above your arm. Once you've gone another 2π, you should find that your hand is in the same, original orientation, but your arm is untwisted! Note the importance of being able to rotate above and below; you can't do that in two-dimensions. In total, you rotated your hand through an angle of 4π (2π twice), and everything gets back to where it started. So apparently,

$4\pi = 0$ for three-dimensional rotations.

This further demonstrates that the set of three-dimensional rotations are simply connected: we can <u>always</u> rotate by 4π and get back to where we started, as if nothing ever happened. By the way, this demonstration is called the "plate trick", "belt trick", and several other names.

On Friday or perhaps after spring break we'll see the consequence of this observation that "$4\pi = 0$". For now, we'll just file it away with the non-Abelian nature of three-dimensional rotations as well. For the rest of this lecture, we'll work to construct the unitary operator that implements these three dimensional rotations. Let's call this operator $\mathbb{U}$, as it is unitary. Further, in three-dimensions, you can rotate about three independent, orthogonal axes: $\hat{x}$, $\hat{y}$, and $\hat{z}$. We'll call the angles we rotate about the appropriate axis θ_x, θ_y, or θ_z and can form a three-dimensional vector from them:

$\vec{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$. Then, $\mathbb{U}$ is a function of $\vec{\theta}$: $\mathbb{U}(\vec{\theta})$.

As always, because $\mathbb{U}$ is unitary, it can be expressed as the exponential of a Hermitian operator $\hat{L}$, where we have:

$$\mathbb{U}(\vec{\theta}) = e^{i\vec{\theta}\cdot\hat{\vec{L}}} = e^{i(\theta_x \hat{L}_x + \theta_y \hat{L}_y + \theta_z \hat{L}_z)},$$

for the three Hermitian operators L_x, L_y, L_z. Actually, we already identified $\hat{L}_z$, as a rotation about the $\hat{z}$-axis is equivalent to rotation of the x-y plane:

$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$. "x", "y", and "z" are just labels, so we can also write down $\hat{L}_x$ and $\hat{L}_y$ by permuting x, y, z:

$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$, $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$.

Note that the x, y, z order is important in these expressions: With the subscript of $\hat{L}$, the first term has x-y-z in that cyclic order. Of course, this is nothing more than our familiar angular momentum, but operator-ized for quantum mechanics. If we define $\hat{\vec{r}}$ and $\hat{\vec{p}}$ the position and momentum vector operators:

$$\hat{\vec{r}} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}, \quad \hat{\vec{p}} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix},$$

then the three-dimensional angular momentum operator $\hat{\vec{L}}$ is:

$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$, the familiar cross product. So, our unitary rotation matrix $\mathbb{U}(\vec{\theta})$ can be written as:

$$\mathbb{U}(\vec{\theta}) = e^{i\hat{\vec{L}}\cdot\hat{\vec{\theta}}} = e^{i\hat{\vec{\theta}}\cdot(\hat{\vec{r}}\times\hat{\vec{p}})}.$$

So, we just need to specify the three numbers of $\vec{\theta}$, and we can perform any rotation we want.

Does the set of all $\mathbb{U}(\vec{\theta})$'s, $\{\mathbb{U}(\vec{\theta})\}$ form a group? Well, let's just check each of the four properties. First, is the identity in this set, $\mathbb{1}$? Well, the identity operator corresponds to no rotation, $\vec{\theta} = \vec{0}$, so this is indeed a "rotation". one property down.

What about the existence of an inverse? For $\mathbb{U}(\vec{\theta})$, what is its inverse? Well, to get back to the identity, we just rotate the opposite direction, by $-\vec{\theta}$. This is still a rotation, so if $\mathbb{U}(\vec{\theta})$ is in the group, then so too is $\mathbb{U}(-\vec{\theta})$. Further, from the unitarity and exponential form, note that:

$$\mathbb{U}(-\vec{\theta}) = e^{i(-\vec{\theta})\cdot\hat{\vec{L}}} = e^{-i\vec{\theta}\cdot\hat{\vec{L}}} = \left(e^{i\vec{\theta}\cdot\hat{\vec{L}}}\right)^{\dagger} = \mathbb{U}(\vec{\theta})^{\dagger},$$

because $\vec{\theta}$ is a real vector and $\hat{\vec{L}}$ is Hermitian. This must have been true because unitarity required:

$$\mathbb{U}(\vec{\theta})\,\mathbb{U}(\vec{\theta})^{\dagger} = \mathbb{1}.$$

Next, what about associativity? Well, we're considering linear operators, so we can always imagine that they are matrices. Matrix multiplication is associative, so we're good there.

The final requirement of a group is <u>closure</u>: for two angle vectors $\vec{\theta}, \vec{\phi}$, then the product of their corresponding unitary rotation operators <u>must</u> be in the group:

$\mathbb{U}(\vec{\theta})\,\mathbb{U}(\vec{\phi})$ is in the group of rotations. Does this impose constraints on the angularmomentum operator $\hat{\vec{L}}$, or, rather on its individual components, $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$? What is the multiplication rule for three-dimensional rotations? Can we express it in generality? More next time...